

# On infinite spectra of first order properties of random graphs

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## 1 Introduction

Asymptotic behavior of first-order properties probabilities of the Erdős–Rényi random graph  $G(n, p)$  have been widely studied in [1]–[3], [7]–[14], [22]. Let  $n \in \mathbb{N}$ ,  $0 \leq p \leq 1$ . Consider a set  $\Omega_n = \{G = (V_n, E)\}$  of all undirected graphs without loops and multiple edges with a set of vertices  $V_n = \{1, 2, \dots, n\}$ . *Erdős–Rényi random graph* [1, 7, 11, 22] is a random element  $G(n, p)$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that it maps  $\Omega$  to  $\Omega_n$  and its distribution  $\mathbb{P}_{n,p}$  on  $\mathcal{F}_n = 2^{\Omega_n}$  is defined in the following way:

$$\mathbb{P}_{n,p}(G) = p^{|E|}(1-p)^{C_n^2-|E|}.$$

Let us denote the event “ $G(n, p)$  follows a property  $L$ ” by  $\{G(n, p) \models L\}$ .

The random graph *obeys Zero-One Law*, if for any first order property  $L$  (see [15]) the probability  $\mathbb{P}(G(n, p) \models L)$  either tends to 0 or tends to 1. In [9], it was proved that if  $p = n^{-\alpha+o(1)}$ ,  $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ , then  $G(n, p)$  obeys Zero-One Law. To avoid trivialities, we shall restrict ourselves to  $0 < \alpha < 1$  (the case  $p = O(1/n)$  was studied in [9]). If  $\alpha \in \mathbb{Q} \cap (0, 1)$ , then  $G(n, n^{-\alpha})$  does not obey Zero-One Law (see, e.g., [22]).

In [16]–[22], Zero-One  $k$ -Law was studied (the random graph obeys *Zero-One  $k$ -Law*, if for any property  $L$  which is expressed by a first-order formula with a quantifier depth at most  $k$  (see [15]) the probability  $\mathbb{P}(G(n, p) \models L)$  either tends to 0 or tends to 1). Let us remind that a *quantifier depth* of a first-order formula is the maximum number of nested quantifiers. We denote a set of all graph properties which are expressed by first order formulae with a quantifier depth at most  $k$  by  $\mathcal{L}_k$ . Moreover, let  $\mathcal{L} = \bigcup_{k \in \mathbb{N}} \mathcal{L}_k$  be the set of all first order graph properties.

In 2012, we proved that if  $k \geq 3$  and  $\alpha \in (0, 1/(k-2))$  (see [20, 21]), then  $G(n, n^{-\alpha})$  obeys Zero-One  $k$ -Law. Moreover, in these papers we proved that  $G(n, n^{-1/(k-2)})$  does not obey Zero-One  $k$ -Law. In 2014 [16], we proved that if  $k > 3$  and  $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$ ,  $\beta \in (0, \infty) \setminus \mathcal{Q}$ , where  $\mathcal{Q}$  is the set of all positive fractions with a numerator at most  $2^{k-1}$ , then  $G(n, n^{-\alpha})$  obeys Zero-One  $k$ -Law. Moreover, in the paper it was proved that  $G(n, n^{-\alpha})$  does not obey Zero-One  $k$ -Law, if  $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$ , where  $\beta \in \{0, 1, \dots, 2^{k-1} - 2\}$ . Finally, in [19] it was proved that  $G(n, n^{-\alpha})$  obeys Zero-One  $k$ -Law, if  $\alpha \in \{1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\}$ . Thus,  $1 - \frac{1}{2^{k-2}}$  — is the maximum  $\alpha$  in  $(0, 1)$  such that  $G(n, n^{-\alpha})$  does not obey Zero-One  $k$ -Law.

In the presented paper, we prove (see Section 2) that in  $(1 - \frac{1}{2^{k-1}}, 1)$  there is only a finite number of  $\alpha$  such that  $G(n, n^{-\alpha})$  does not obey Zero-One  $k$ -Law.

If the random graph  $G(n, n^{-\alpha})$  does not obey Zero-One  $k$ -Law for some  $\alpha \in (0, 1)$  and  $k \in \mathbb{N}$ , then we say that  $\alpha$  is in a *spectrum of  $k$* . Let us remind that in [14] two notions of spectra of a first-order property  $L \in \mathcal{L}$  were considered. The first considers  $p = n^{-\alpha}$ .  $S^1(L)$  is a set of  $\alpha \in (0, 1)$  which does *not* satisfy the following property: With  $p(n) = n^{-\alpha}$ ,  $\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p(n)) \models L)$  exists and is either zero or one. The second considers  $p = n^{-\alpha+o(1)}$ .  $S^2(L)$  is a set of  $\alpha \in (0, 1)$  which does *not* satisfy the following property: There exists  $\delta \in \{0, 1\}$  and  $\epsilon > 0$  so that when  $n^{-\alpha-\epsilon} < p(n) < n^{-\alpha+\epsilon}$ ,  $\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p(n)) \models L) = \delta$ . Let  $k \in \mathbb{N}$ . Denote unions of  $S^1(L)$  and  $S^2(L)$  over all  $L \in \mathcal{L}_k$  by  $S_k^1$  and  $S_k^2$  respectively.

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In [10], it was proved that the sets  $S_k^1$  and  $S_k^2$  are infinite when  $k$  is large enough. There are, up to tautological equivalence, (see, e.g., [15]) only a finite number of first order sentences with a given quantifier depth. Thus, for  $j$  either 1 or 2, the set  $S_k^j$  is infinite if and only if there is a single  $L$  with quantifier depth at most  $k$  such that  $S^j(L)$  is infinite. Therefore, we always search for one property with an infinite spectrum when we prove that the spectrum  $S_k^j$  is infinite.

It is also known [13] that all limit points of  $S_k^1$  and  $S_k^2$  are approached only from above.

In [14], it was proved that the minimum  $k_1$  and  $k_2$  such that the sets  $S_{k_1}^1$  and  $S_{k_2}^2$  are infinite are in the sets  $\{4, \dots, 12\}$  and  $\{4, \dots, 10\}$  respectively. Moreover, in the same paper we estimate the minimum and the maximum limit points of  $S_k^1$ ,  $S_k^2$ . Denote sets of limit points of  $S_k^1$  and  $S_k^2$  by  $(S_k^1)'$  and  $(S_k^2)'$  respectively. Then

$$\min(S_k^1)' \in \left[ \frac{1}{k-2}, \frac{1}{k-11} \right], \text{ if } k \geq 15, \quad \min(S_k^2)' \in \left[ \frac{1}{k-1}, \frac{1}{k-7} \right], \text{ if } k \geq 10,$$

$$\max(S_k^j)' \in \left[ 1 - \frac{1}{2^{k-13}}, 1 - \frac{1}{2^{k-1}} \right], \quad \text{if } k \geq 16, j \in \{1, 2\}.$$

In the next section, we state new results. We prove them in Section 4. Some statements on a distribution of small subgraphs in the random graph, which were used in our proofs, are formulated in Section 3.

## 2 New results

**Theorem 1** *For any  $k \geq 5$ ,  $\frac{1}{\lfloor k/2 \rfloor} \in (S_k^1)'$ .*

So, we obtain a better upper bound on the minimum limit point of  $S_k^1$  for any  $k \leq 20$  and a better upper bound on the minimum limit point of  $S_k^2$  for all  $k \leq 12$ . Moreover, Theorem 1 and Zero-One  $k$ -Law from [20, 21] imply the following statement.

**Corollary 1** *The minimum  $k$  such that the set  $S_k^1$  ( $S_k^2$ ) is infinite equals 4 or 5.*

Moreover, we obtain a better lower bound on the maximum limit point of spectra (for small  $k$  as well).

**Theorem 2** *For any  $k \geq 8$ ,  $1 - \frac{1}{2^{k-5}} \in (S_k^1)'$ .*

An emptiness of an intersection of  $S_k^1$  with  $(1 - \frac{1}{2^{k-1}}, 1)$  follows from the result, which is stated below.

**Theorem 3** *Let  $k > 3$ ,  $b$  be arbitrary natural numbers. Moreover, let  $\frac{a}{b}$  be an irreducible positive fraction. Denote  $\nu = \max\{1, 2^{k-1} - b\}$ . Let  $a \in \{\nu, \nu + 1, \dots, 2^{k-1}\}$ ,  $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$ . Then the random graph  $G(n, n^{-\alpha})$  obeys Zero-One  $k$ -Law.*

### 3 Small subgraphs in the random graph

For an arbitrary graph  $G = (E, V)$ , set  $e(G) = |E|$ ,  $v(G) = |V|$ ,  $\rho(G) = \frac{e(G)}{v(G)}$ ,  $\rho^{\max}(G) = \max_{H \subseteq G} \rho(H)$  ( $\rho(G)$  is called a *density of G*). Denote the number of copies of  $G$  in  $G(n, p)$  by  $N_G$ . Denote the property of containing a copy of  $G$  by  $L_G$ .

**Theorem 4 ([2, 8])** *If  $p = o(n^{-1/\rho^{\max}(G)})$ , then  $\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \models L_G) = 0$ . If  $n^{-1/\rho^{\max}(G)} = o(p)$ , then  $\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \models L_G) = 1$ .*

In other words, the function  $n^{-1/\rho^{\max}(G)}$  is a *threshold* (see [1, 7]) for the property  $L_G$ .

Let  $G$  be a *strictly balanced graph* (a density of this graph is greater than a density of any its proper subgraph) with  $a(G)$  automorphisms.

**Theorem 5 ([2])** *If  $p = n^{-1/\rho(G)}$ , then  $N_G \xrightarrow{d} \text{Pois}\left(\frac{1}{a(G)}\right)$ .*

Consider arbitrary graphs  $G$  and  $H$  such that  $H \subset G$ ,  $V(H) = \{x_1, \dots, x_m\}$ ,  $V(G) = \{x_1, \dots, x_l\}$  and the set  $E(G) \setminus (E(H) \cup E(G \setminus H))$  is non-empty. Denote  $e(G, H) = e(G) - e(H)$ ,  $v(G, H) = v(G) - v(H)$ ,  $\rho(G, H) = \frac{e(G, H)}{v(G, H)}$ ,  $\rho^{\max}(G, H) = \max_{H \subset K \subseteq G} \rho(K, H)$ . Moreover, let  $e^{\min}(G, H)$  be the minimum number  $e(K, H)$  over all graphs  $K$  such that  $H \subset K \subseteq G$ ,  $\rho(K, H) = \rho^{\max}(G, H)$  and the set  $E(K) \setminus (E(H) \cup E(K \setminus H))$  is non-empty. Consider graphs  $\tilde{H}$ ,  $\tilde{G}$ , where  $V(\tilde{H}) = \{\tilde{x}_1, \dots, \tilde{x}_m\}$ ,  $V(\tilde{G}) = \{\tilde{x}_1, \dots, \tilde{x}_l\}$ ,  $\tilde{H} \subset \tilde{G}$ . The graph  $\tilde{G}$  is called  $(G, (x_1, \dots, x_m))$ -*extension of the ordered tuple*  $(\tilde{x}_1, \dots, \tilde{x}_m)$ , if

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Rightarrow \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\} \in E(\tilde{G}) \setminus E(\tilde{H}).$$

The extension is called *strict*, if

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Leftrightarrow \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\} \in E(\tilde{G}) \setminus E(\tilde{H}).$$

Denote the property of containing a  $(G, (x_1, \dots, x_m))$ -extension of any ordered tuple of  $m$  vertices by  $L_{(G, H)}$ .

**Theorem 6 ([12])** *There exists  $0 < \varepsilon < K$  such that*

$$\text{if } p \leq \varepsilon n^{-1/\rho^{\max}(G, H)} (\ln n)^{1/e^{\min}(G, H)}, \text{ then } \lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \models L_{(G, H)}) = 0;$$

$$\text{if } p \geq K n^{-1/\rho^{\max}(G, H)} (\ln n)^{1/e^{\min}(G, H)}, \text{ then } \lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \models L_{(G, H)}) = 1.$$

Obviously, for a *balanced pair*  $(G, H)$  (the maximum density  $\rho^{\max}(G, H)$  equals  $\rho(G, H)$ ) the quantity  $\rho^{\max}(G, H)$  in the statement of Theorem 6 can be replaced by  $\rho(G, H)$ . In the same way as for graphs, the pair  $(G, H)$  is called *strictly balanced*, if  $\rho(G, H) > \rho(K, H)$  for any graph  $K$  such that  $H \subset K \subset G$ .

Fix a number  $\alpha \in (0, 1)$ . Set

$$v(G, H) = |V(G) \setminus V(H)|, \quad e(G, H) = |E(G) \setminus E(H)|,$$

$$f_\alpha(G, H) = v(G, H) - \alpha e(G, H).$$

If for any graph  $S$  such that  $H \subset S \subseteq G$  the inequality  $f_\alpha(S, H) > 0$  holds, then the pair  $(G, H)$  is called  $\alpha$ -safe (see [7, 22]). Finally, let us introduce a notion of a maximal pair. Let  $\tilde{H} \subset \tilde{G} \subset \Gamma$  and  $T \subset K$ , where  $V(T) = \{v_1, \dots, v_t\}$ ,  $t \leq |V(\tilde{G})|$ . The pair  $(\tilde{G}, \tilde{H})$  is called  $(K, T)$ -maximal in  $\Gamma$ , if any ordered tuple  $\mathbf{t}$  of  $t$  vertices from  $V(\tilde{G})$  with at least one vertex from  $V(\tilde{G}) \setminus V(\tilde{H})$  does not have a strict  $(K, (v_1, \dots, v_t))$ -extension  $\tilde{K}$  in  $\Gamma$  such that the following properties hold. The intersection of the sets  $V(\tilde{K})$ ,  $V(\tilde{G})$  contains vertices from  $\mathbf{t}$  only and any vertex from  $V(\tilde{K})$  which is not in  $\mathbf{t}$  and any vertex from  $V(\tilde{G})$  which is not in  $\mathbf{t}$  are not adjacent. Similarly, the graph  $\tilde{G}$  is called  $(K, T)$ -maximal in  $\Gamma$ , if any ordered tuple  $\mathbf{t}$  of  $t$  vertices from  $V(\tilde{G})$  does not have a strict  $(K, (v_1, \dots, v_t))$ -extension  $\tilde{K}$  in  $\Gamma$  such that the following properties hold. The intersection of the sets  $V(\tilde{K})$ ,  $V(\tilde{G})$  contains vertices from  $\mathbf{t}$  only and any vertex from  $V(\tilde{K})$  which is not in  $\mathbf{t}$  and any vertex from  $V(\tilde{G})$  which is not in  $\mathbf{t}$  are not adjacent.

Consider the random graph  $G(n, p)$ , arbitrary vertices  $\tilde{x}_1, \dots, \tilde{x}_m \in V_n$  and a random variable  $N_{(G, H)}^{(K, T)}(\tilde{x}_1, \dots, \tilde{x}_m)$  that maps each graph  $\mathcal{G}$  from  $\Omega_n$  to the number of strict  $(G, (x_1, \dots, x_m))$ -extensions  $\tilde{G}$  of  $(\tilde{x}_1, \dots, \tilde{x}_m)$  in  $\mathcal{G}$  such that the pair  $(\tilde{G}, \tilde{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_m\}})$  is  $(K, T)$ -maximal in  $\mathcal{G}$  (and  $N_{(G, H)}(\tilde{x}_1, \dots, \tilde{x}_m)$  is the number of all  $(G, (x_1, \dots, x_m))$ -extensions of  $(\tilde{x}_1, \dots, \tilde{x}_m)$  in  $\mathcal{G}$ ). Let us state the result, which was proved in [14], on an asymptotic behavior of this variable.

**Theorem 7 ([14])** *Let  $0 < \alpha_1 < \alpha_2 < 1$ . Let a pair  $(G, H)$  be  $\alpha_2$ -safe,  $f_{\alpha_1}(K, T) < 0$  and  $v(T) \leq v(G)$ . Let  $p \in [n^{-\alpha_2}, n^{-\alpha_1}]$ . Then a.a.s. for any  $\tilde{x}_1, \dots, \tilde{x}_m$  the inequality  $N_{(G, H)}^{(K, T)}(\tilde{x}_1, \dots, \tilde{x}_m) > 0$  holds.*

If  $\tilde{H} = (\emptyset, \emptyset)$  and  $(\tilde{G}, \tilde{H})$  is  $(K, T)$ -maximal in  $\Gamma$ , then  $\tilde{G}$  is  $(K, T)$ -maximal in  $\Gamma$ . Therefore, we can state a particular case of Theorem 7 which considers  $(K, T)$ -maximal graphs. Let  $N_G^{(K, T)}$  be a random variable that maps each  $\mathcal{G}$  from  $\Omega_n$  to the number of  $(K, T)$ -maximal copies of  $G$  in  $\mathcal{G}$ .

**Corollary 2** *Let  $0 < \alpha_1 < \alpha_2 < 1$ . Let  $G$  be a strictly balanced graph with  $\rho(G) < 1/\alpha_2$  and  $f_{\alpha_1}(K, T) < 0$ . If  $p \in [n^{-\alpha_2}, n^{-\alpha_1}]$ , then a.a.s. the inequality  $N_G^{(K, T)} > 0$  holds.*

Let us call pairs  $(G, (x_1, \dots, x_m))$  and  $(\tilde{G}, (\tilde{x}_1, \dots, \tilde{x}_m))$ , where  $\{x_1, \dots, x_m\} \subset V(G)$  and  $\{\tilde{x}_1, \dots, \tilde{x}_m\} \subset \tilde{V}(G)$ , *isomorphic*, if the graph  $\tilde{G}$  is a strict  $(G, (x_1, \dots, x_m))$ -extension of  $(\tilde{x}_1, \dots, \tilde{x}_m)$ .

Moreover, in our proofs we use a lemma on the existence of a copy of a strictly balanced graph without extensions, which is stated below. A method for obtaining such results is introduced in [3]. Here, we use this method to prove the lemma.

Let  $H$  be a strictly balanced graph,  $(G, H)$  be a strictly balanced pair,  $\rho(H) = \rho(G, H) = 1/\alpha$ . Moreover, let  $V(H) = \{h_1, \dots, h_v\}$ , where  $v = v(H)$ . Let  $W$  be a set with the maximum cardinality which contains ordered tuples of  $v$  vertices from  $V_n$  which satisfy the following property. For any two ordered tuples  $w_1 = (x_{i_1}, \dots, x_{i_v}), w_2 = (x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(v)}}) \in W$  which coincide as sets, a permutation  $\sigma$  of  $(h_1, \dots, h_v)$  does not preserve edges of  $H$  (i.e. a mapping  $\phi : V(H) \rightarrow V(H)$  such that  $\phi(h_i) = h_{\sigma(i)}$ ,  $i \in \{1, \dots, v\}$ , is not an automorphism of  $H$ ). Obviously,  $|W| = \frac{n!}{(n-v)!a(H)}$ . For any  $w \in W$ , we denote a set of elements of  $w$  by  $\overline{w}$ . For any  $w = (x_{i_1}, \dots, x_{i_v}) \in W$ , consider an event  $A_w$  that some spanning subgraph in  $G(n, n^{-\alpha})|_{\overline{w}}$  is isomorphic to  $H$  and the corresponding isomorphism maps  $x_{i_j}$  to  $h_j$  for each  $j \in \{1, \dots, v\}$ .

**Lemma 1** *There exists a subsequence  $\{n_i\}_{i \in \mathbb{N}}$  of the sequence of positive integers such that the following property holds. With positive asymptotic probability less than 1, in  $G(n_i, n_i^{-\alpha})$  there exists*

at least one copy of  $H$  and for any  $w \in V_{n_i}$  either  $\overline{A_w}$  holds or there is no  $(G, (h_1, \dots, h_v))$ -extension of  $w$  in  $G(n_i, n_i^{-\alpha})$ .

*Proof.* Denote  $N_H^-(w) = \sum_{\tilde{w}} I(A_{\tilde{w}})$ , where the summation is taken over all  $\tilde{w} \in W$  which do not intersect  $w$ . Denote  $N_H^+(w) = \sum_{\tilde{w}} \xi_{\tilde{w}}$ , where the summation is taken over all  $\tilde{w} \in W$  which intersect  $w$  such that  $\overline{\tilde{w}} \cap \overline{w} \neq \overline{w}$ . The random variable  $\xi_{\tilde{w}}$  is defined in the following way. For any  $\mathcal{G} \in \Omega_n$ , the equality  $\xi_{\tilde{w}}(\mathcal{G}) = 1$  holds if and only if  $\mathcal{G}$  with edges between any two vertices from  $\overline{w} \cap \overline{\tilde{w}}$  follows  $A_w$  (otherwise,  $\xi_{\tilde{w}}(\mathcal{G}) = 0$ ). Set  $N_H(w) = N_H^-(w) + N_H^+(w)$ .

Denote a probability of the event that in  $G(n, n^{-\alpha})$  there exists at least one copy of  $H$  and for any ordered tuple  $w$  of  $v$  vertices from  $V_n$  either  $A_w$  holds or there is no  $(G, (h_1, \dots, h_v))$ -extension of  $w$  by  $\mu_n$ . Then

$$\mathbb{P}(N_H > 0) \geq \mu_n = \mathbb{P}(N_G = 0) - \mathbb{P}(N_H = 0) \geq \mathbb{P}(N_H = 1, N_G = 0).$$

Theorem 5 implies  $\lim_{n \rightarrow \infty} \mathbb{P}(N_H > 0) = 1 - e^{-1/a(H)}$ . Finally,

$$\begin{aligned} \mathbb{P}(N_H = 1, N_G = 0) &= \sum_{w \in W} \mathbb{P}(N_H = 1, N_G = 0 | A_w) \mathbb{P}(A_w) = \\ &= \sum_{w \in W} \mathbb{P}(N_H(w) = 0, N_{(G,H)}(w) = 0 | A_w) \mathbb{P}(A_w) = \sum_{w \in W} \mathbb{P}(N_H(w) = 0, N_{(G,H)}(w) = 0) \mathbb{P}(A_w) = \\ &= \mathbb{P}(N_H(w_0) = 0, N_{(G,H)}(w_0) = 0) \sum_{w \in W} \mathbb{P}(I_w) \sim \frac{1}{a(H)} \mathbb{P}(N_H^-(w_0) = 0, N_{(G,H)}(w_0) = 0), \end{aligned}$$

where  $w_0 \in W$  is an arbitrary ordered tuple. Asymptotic equality holds, because Theorem 4 implies that a.s. in  $G(n, n^{-\alpha})$  there does not exist any subgraph with at most  $2v$  vertices and a density greater than  $1/\alpha$ . The probability  $\mathbb{P}(N_H^-(w_0) = 0, N_{(G,H)}(w_0) = 0)$  converges to some positive number which is less than 1 (see [3]). Therefore, the lemma is proved.

## 4 Proofs

First of all, let us introduce some notations.

Let  $\mathcal{G}$  be an arbitrary graph. Moreover, let  $r, s$  be arbitrary natural numbers. For any vertices  $x_1, \dots, x_s$  of  $\mathcal{G}$ , we denote a set of all common  $r$ -neighbors of  $x_1, \dots, x_s$  in  $\mathcal{G}$  by  $N_r(x_1, \dots, x_s)$  (we omit  $\mathcal{G}$  in this notation when there is no risk of confusion). A  $r$ -neighbor of a vertex  $x$  is a vertex  $y$  such that the minimum length of a path which connects  $x$  and  $y$  equals  $r$  (a *length of a path* is a number of edges in it). Set  $N(x_1, \dots, x_s) := N_1(x_1, \dots, x_s)$ .

Moreover, for any two arbitrary vertices  $x, y$  of  $\mathcal{G}$  and any its subgraphs  $A, B$  denote a length of a minimal path in  $\mathcal{G}$  which connects  $x$  and  $y$  by  $d_{\mathcal{G}}(x, y)$  (a *minimal path* is a path with the minimum length among considered paths). Moreover, we call a path which connects  $x$  and some vertex of  $A$  a *minimal path which connects  $x$  and  $A$  in  $\mathcal{G}$*  if its length equals  $\min_{y \in B} d_{\mathcal{G}}(x, y)$ . Set  $d_{\mathcal{G}}(x, A) = d_{\mathcal{G}}(A, x) = \min_{v \in V(A)} d_{\mathcal{G}}(x, v)$ ,  $d_{\mathcal{G}}(A, B) = \min_{v \in V(A)} d_{\mathcal{G}}(v, B)$ .

### 4.1 Proof of Theorem 1

Let  $k \geq 5$ ,  $m \in \mathbb{N}$ ,  $\alpha = \frac{1}{\lfloor k/2 \rfloor} + \frac{1}{\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1)}$  and  $p = n^{-\alpha}$ .

Consider a set  $\tilde{\Omega}_n$  of all graphs  $\mathcal{G}$  from  $\Omega_n$  which follow the properties below.

1. For any strictly balanced pair  $(G, H)$  such that  $V(H) = \{h_1, \dots, h_v\}$ ,  $\rho(G, H) < 1/\alpha$ ,  $v \leq m + \lfloor k/2 \rfloor - 1$ ,  $v(G) \leq 2(m + \lfloor k/2 \rfloor - 1)$ , any ordered tuple of  $v$  vertices has a  $(G, (h_1, \dots, h_v))$ -extension in  $\mathcal{G}$ .
2. For any  $G$  with  $v(G) \leq 2(m + \lfloor k/2 \rfloor + 1)$  and  $\rho^{\max}(G) > 1/\alpha$ , in  $\mathcal{G}$  there is no copy of  $G$ .

Theorem 4 and Theorem 6 imply that  $\mathbf{P}(G(n, p) \in \tilde{\Omega}_n) \rightarrow 1$  as  $n \rightarrow \infty$ .

Let  $L$  be a first-order property which is expressed by the formula  $\exists x_1 \dots \exists x_{\lfloor k/2 \rfloor} \varphi(x_1, \dots, x_{\lfloor k/2 \rfloor})$  with the quantifier depth  $\max(2\lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 3) \leq k$ , where  $\varphi(x_1, \dots, x_{\lfloor k/2 \rfloor}) =$

$$[K(x_1, \dots, x_{\lfloor k/2 \rfloor}) \wedge (\exists y_1 \dots \exists y_{\lfloor k/2 \rfloor} [(y_1 \in N(x_1, \dots, x_{\lfloor k/2 \rfloor})) \wedge \dots \wedge (y_{\lfloor k/2 \rfloor} \in N(x_1, \dots, x_{\lfloor k/2 \rfloor})) \wedge K(y_1, \dots, y_{\lfloor k/2 \rfloor})]) \wedge (\neg(\exists z [R_z^2 \wedge \dots \wedge R_z^{\lfloor k/2 \rfloor} \wedge (\forall y ((y \in N(x_1, \dots, x_{\lfloor k/2 \rfloor})) \Rightarrow R_z^{1,2}(y)))]))].$$

Here, we use the following notations:

$$K(x_1, \dots, x_{\lfloor k/2 \rfloor}) = ((x_1 \sim x_2) \wedge (x_1 \sim x_3) \wedge \dots \wedge (x_1 \sim x_{\lfloor k/2 \rfloor}) \wedge \dots \wedge (x_{\lfloor k/2 \rfloor - 1} \sim x_{\lfloor k/2 \rfloor})),$$

$$(y \in N(x_1, \dots, x_{\lfloor k/2 \rfloor})) = ((y \sim x_1) \wedge \dots \wedge (y \sim x_{\lfloor k/2 \rfloor})).$$

For any  $1 \leq i < j \leq \lfloor k/2 \rfloor$ ,

$$R_z^{i,j}(a) = (\exists v [(v \in N(z, a, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{\lfloor k/2 \rfloor})) \wedge (v \nsim x_i) \wedge (v \nsim x_j)]).$$

For any  $2 \leq i \leq \lfloor k/2 \rfloor$ ,

$$R_z^i = (\exists v [v \in N(z, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{\lfloor k/2 \rfloor})]).$$

Suppose that  $\mathcal{G} \in \tilde{\Omega}_n$  follows  $L$ . Consider vertices  $x_1, \dots, x_{\lfloor k/2 \rfloor}$  such that  $\varphi(x_1, \dots, x_{\lfloor k/2 \rfloor})$  is true. Set  $X = \mathcal{G}|_{\{x_1, \dots, x_{\lfloor k/2 \rfloor}\} \cup N(x_1, \dots, x_{\lfloor k/2 \rfloor})}$ ,  $\chi = |V(X)| - \lfloor k/2 \rfloor$ , where  $N(x_1, \dots, x_{\lfloor k/2 \rfloor}) = \{x^1, \dots, x^\chi\}$ . Let us prove that  $\chi \geq m$ . Suppose that  $\chi < m$ . By the definition of  $\tilde{\Omega}_n$ , in  $\mathcal{G}$  there are vertices  $z, v_1, \dots, v_{\chi + \lfloor k/2 \rfloor - 1}$  such that for any  $i \in \{1, \dots, \chi\}$  we have  $v_i \in N(x^i, z, x_3, \dots, x_{\lfloor k/2 \rfloor})$ , and for any  $i \in \{\chi + 1, \dots, \chi + \lfloor k/2 \rfloor - 1\}$  we have  $v_i \in N(z, x_1, \dots, x_{i-\chi}, x_{i-\chi+2}, \dots, x_{\lfloor k/2 \rfloor})$ . Indeed, in this case the pair  $(\mathcal{G}|_{\{x_1, \dots, x_{\lfloor k/2 \rfloor}, v_1, \dots, v_{\chi + \lfloor k/2 \rfloor - 1}, z\} \cup N(x_1, \dots, x_{\lfloor k/2 \rfloor})}, X)$  is strictly balanced with the density

$$\frac{\lfloor k/2 \rfloor(\chi + \lfloor k/2 \rfloor - 1)}{\chi + \lfloor k/2 \rfloor} = \frac{1}{1/\lfloor k/2 \rfloor + 1/(\lfloor k/2 \rfloor(\chi + \lfloor k/2 \rfloor - 1))} < \frac{1}{\alpha}.$$

This contradicts the property  $L$ . Therefore,  $\chi \geq m$ . Now, let us prove that  $\chi = m$ . Suppose  $\chi > m$ . Remove from the set  $N(x_1, \dots, x_{\lfloor k/2 \rfloor})$  some vertices in such a way that  $m + 1$  vertices are in the remainder (but  $\lfloor k/2 \rfloor$  pairwise adjacent vertices are still in the set). Denote a subgraph in  $X$  induced by the union of this remainder with  $x_1, \dots, x_{\lfloor k/2 \rfloor}$  by  $\tilde{X}$ . Then

$$\rho(\tilde{X}) \geq \frac{\lfloor k/2 \rfloor(m + 1) + \lfloor k/2 \rfloor(\lfloor k/2 \rfloor - 1)}{m + 1 + \lfloor k/2 \rfloor} > 1/\alpha.$$

This contradicts Property 2 in the definition of  $\tilde{\Omega}_n$ .

So,  $\chi = m$ . Let  $z$  be a vertex such that the predicate  $R_z^{1,2}$  is true for all vertices from  $N(x_1, \dots, x_{\lfloor k/2 \rfloor})$ , the predicate  $R_z^i$  is true for any  $i \in \{2, \dots, \lfloor k/2 \rfloor\}$ . Then in  $\mathcal{G}$  there exist vertices  $v_1, \dots, v_j$  such that  $z \in N(v_1, \dots, v_j)$  and the set  $\{x_1, \dots, x_{\lfloor k/2 \rfloor - 1}\} \cup N(x_1, \dots, x_{\lfloor k/2 \rfloor})$  can be divided into  $j$  subsets  $N_1, \dots, N_j$  in the following way: for any  $i \in \{1, \dots, j\}$  and any vertex  $y \in N_i$ ,  $y \sim v_i$  and  $v_i$  is

adjacent to  $\lfloor k/2 \rfloor - 2$  vertices from  $\{x_1, \dots, x_{\lfloor k/2 \rfloor}\} \setminus \{y\}$ . Set  $Y = \mathcal{G}|_{\{x_1, \dots, x_{\lfloor k/2 \rfloor}, v_1, \dots, v_j, z\} \cup N(x_1, \dots, x_{\lfloor k/2 \rfloor})}$ . Then

$$1/\rho(Y) \leq \frac{\lfloor k/2 \rfloor + j + 1 + m}{\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1) + m + \lfloor k/2 \rfloor - 1 + j(\lfloor k/2 \rfloor - 1)}.$$

Note that the inequality  $j < m + \lfloor k/2 \rfloor - 1$  implies  $1/\rho(Y) < \alpha$ . Thus, from the definition of  $\tilde{\Omega}_n$  it follows that  $j \geq m + \lfloor k/2 \rfloor - 1$ . As  $j \leq m + \lfloor k/2 \rfloor - 1$ , the equality  $j = m + \lfloor k/2 \rfloor - 1$  holds. Therefore,  $1/\rho(Y) \leq \frac{2(m + \lfloor k/2 \rfloor)}{2\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1)} = \frac{1}{\lfloor k/2 \rfloor} + \frac{1}{\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1)} = \alpha$ ,  $1/\rho(X) \leq \frac{m + \lfloor k/2 \rfloor}{\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1)} = \alpha$ . Property 2 in the definition of  $\tilde{\Omega}_n$  implies equalities  $\rho(X) = \rho(Y) = 1/\alpha$ . As in  $\mathcal{G}$  there is no vertex  $z$ , which follows the above properties, the graph  $\mathcal{G}$  does not contain a copy  $Y$ , which, in turn, contains  $X$ .

In the remaining part of the proof, we will use these notations  $X$  and  $Y$  for the obtained graphs (the first one is strictly balanced, the second one is balanced, the pair  $(Y, X)$  is strictly balanced) with the density  $1/\alpha$ . Moreover, denote the obtained property of  $\mathcal{G}$  (the existence of a copy of  $X$  such that no copy of  $Y$  contains it) by  $\tilde{L}$ . So, we have proved that if  $\mathcal{G} \in \tilde{\Omega}_n$  and  $\mathcal{G}$  follows  $L$ , then  $\mathcal{G}$  follows  $\tilde{L}$ .

Suppose that  $\mathcal{G} \in \tilde{\Omega}_n$  and  $\mathcal{G}$  follows  $\tilde{L}$ . Obviously, in this case  $\mathcal{G}$  follows  $L$  as well.

By Lemma 1, there exists a partial limit  $\lim_{i \rightarrow \infty} \mathbb{P}(G(n_i, n_i^{-\alpha}) \models \tilde{L}) = c$ , which is not 0 or 1. Moreover,

$$\begin{aligned} \mathbb{P}(G(n_i, n_i^{-\alpha}) \models L) &\sim \mathbb{P}(G(n_i, n_i^{-\alpha}) \in \tilde{\Omega}_{n_i}, G(n_i, n_i^{-\alpha}) \models L) = \\ &= \mathbb{P}(G(n_i, n_i^{-\alpha}) \in \tilde{\Omega}_{n_i}, G(n_i, n_i^{-\alpha}) \models \tilde{L}) \sim \mathbb{P}(G(n_i, n_i^{-\alpha}) \models \tilde{L}) = c. \end{aligned} \quad (1)$$

Since  $\frac{1}{\lfloor k/2 \rfloor} + \frac{1}{\lfloor k/2 \rfloor(m + \lfloor k/2 \rfloor - 1)} \rightarrow \frac{1}{\lfloor k/2 \rfloor}$  as  $m \rightarrow \infty$ , the theorem is proved.

## 4.2 Proof of Theorem 2

Let  $m \geq 2$  be arbitrary natural numbers,  $\alpha = 1 - \frac{1}{2^{k-5}} + \frac{1}{2^{k-5}m}$  and  $p = n^{-\alpha}$ .

Consider a set  $\tilde{\Omega}_n$  of all graphs  $\mathcal{G}$  from  $\Omega_n$  which follow the properties below.

1. For any strictly balanced pair  $(G, H)$  such that  $V(H) = \{h_1, \dots, h_v\}$ ,  $\rho(G, H) < 1/\alpha$ ,  $v \leq (2^{k-5} - 1)(m - 1) + 2$ ,  $v(G) \leq 2(2^{k-5} - 1)(m - 1) + 3$ , any ordered tuple of  $v$  vertices has a  $(G, (h_1, \dots, h_v))$ -extension in  $\mathcal{G}$ .
2. For any  $G$  with  $v(G) \leq 2(2^{k-5} - 1)(m + 1) + 2$  and  $\rho^{\max} > 1/\alpha$ , in  $\mathcal{G}$  there is no copy of  $G$ .

Theorem 4 and Theorem 6 imply that  $\mathbb{P}(G(n, p) \in \tilde{\Omega}_n) \rightarrow 1$  as  $n \rightarrow \infty$ .

The property of vertices  $x$  and  $y$  to be at the distance  $i$  (i.e., a length of the minimal path which connects  $x$  and  $y$  equals  $i$ ) is expressed by the following formula:

$$D_i^*(x, y) = D_i(x, y) \wedge \left( \neg \left( \bigvee_{j=1}^{i-1} D_j(x, y) \right) \right),$$

where  $D_i(x, y)$  — is the following formula with the quantifier depth  $\lceil \log_2 i \rceil$ :

$$D_i(x, y) = \exists v (D_{i/2}(x, v) \wedge D_{i/2}(y, v)), \text{ if } i \text{ is even,}$$

$$D_i(x, y) = \exists v (D_{(i-1)/2}(x, v) \wedge D_{(i+1)/2}(y, v)), \text{ if } i \text{ is odd,}$$

and  $D_1(x, y) = (x \sim y)$ ,  $D_0(x, y) = (x = y)$ . Moreover, set  $D_{i,j}^*(x, y, z) = D_i^*(x, z) \wedge D_j^*(z, y)$ .

Let  $L$  be a first-order property which is expressed by the formula  $\exists a \exists b \varphi(a, b)$  with the quantifier depth  $k$ , where  $\varphi(a, b) =$

$$(S(a, b) \wedge [\forall u (D_{2^{k-6}, 2^{k-6}}^*(a, b, u) \Rightarrow R(a, u))] \wedge [\neg(\exists z ((z \neq a) \wedge (\forall u (D_{2^{k-6}, 2^{k-6}}^*(a, b, u) \Rightarrow D_{2^{k-5}}^*(u, z)))))]).$$

The predicate  $S(a, b) =$

$$(D_{2^{k-5}}^*(a, b) \wedge (\neg(\exists u_1 \exists u_2 \exists x [(u_1 \neq u_2) \wedge D_{2^{k-6}, 2^{k-6}}^*(u_1, u_2, b) \wedge D_{2^{k-6}, 2^{k-6}}^*(u_1, u_2, a) \wedge \psi(a, b, u_1, u_2, x)]))),$$

where  $\psi(a, b, u_1, u_2, x) =$

$$\left( \neg \left( \left( \bigvee_{s=2^{k-6}}^{2^{k-5}} \bigvee_{i=1}^s \bigvee_{j=2^{k-6}-i}^{2^{k-5}-i} (D_{i, s-i}^*(a, u_1, x) \wedge D_j^*(x, u_2)) \right) \vee \left( \bigvee_{i=1}^{2^{k-6}} (D_{i, 2^{k-6}-i}^*(u_1, b, x) \wedge D_i^*(u_2, x)) \right) \right) \right)$$

is true when there do not exist two distinct paths with lengths at most  $2^{k-5}$  which connect the vertex  $a$  and two distinct vertices from the set  $N_{2^{k-6}}(a, b)$  (moreover, any two distinct vertices from  $N_{2^{k-6}}(a, b)$  do not have common neighbors) and there do not exist two distinct intersecting paths with length  $2^{k-6}$  which connect the vertex  $b$  and two distinct vertices from the set  $N_{2^{k-6}}(a, b)$ . The truth of the predicate  $R(a, u) =$

$$(\exists x_1 \exists x_2 [D_{2^{k-6}, 2^{k-6}}^*(a, u, x_1) \wedge D_{2^{k-7}, 2^{k-7}}^*(a, u, x_2) \wedge (\neg D_{2^{k-7}}^*(x_1, x_2)) \wedge \xi(a, x_1, x_2) \wedge \xi(u, x_1, x_2)]),$$

$$\xi(a, x_1, x_2) = \left( \neg \left( \exists y \left( \bigvee_{i=1}^{2^{k-7}-1} (D_{i, 2^{k-6}-i}^*(a, x_1, y) \wedge D_{2^{k-7}-i}^*(y, x_2)) \right) \right) \right),$$

implies the existence of two non-intersecting paths with lengths  $2^{k-6}$  and  $2^{k-5}$  which connect the vertex  $a$  and  $u$ .

Suppose that a graph  $\mathcal{G} \in \tilde{\Omega}_n$  follows  $L$ . Let  $a, b$  be vertices such that the formula  $\varphi(a, b)$  is true. Let  $X$  be a union of all paths with length  $2^{k-5}$  which connect  $a$  and  $b$  in  $\mathcal{G}$ . Let  $\chi$  be a number of all such paths and  $N_{2^{k-6}}(a, b) = \{x^1, \dots, x^\chi\}$ . Let us prove that  $\chi \geq m$ . Suppose that  $\chi < m$ . By the definition of  $\tilde{\Omega}_n$ , in  $\mathcal{G}$  there exists a vertex  $z$  such that for any  $i \in \{1, \dots, \chi\}$  the property  $D_{2^{k-5}}^*(x^i, z)$  holds and there exist  $\chi$  paths  $P_1, \dots, P_\chi$  with length  $2^{k-5}$  connecting  $z$  and  $x^1, \dots, x^\chi$  respectively such that for any distinct  $i, j \in \{1, \dots, \chi\}$  equality  $V(P_i) \cap V(P_j) = \{z\}$  holds. Indeed, if these paths exist, then the pair  $(X \cup P_1 \cup \dots \cup P_\chi, X)$  is strictly balanced and its density equals

$$\frac{2^{k-5}\chi}{(2^{k-5}-1)\chi+1} = \frac{1}{1-1/2^{k-5}+1/(\chi 2^{k-5})} < \frac{1}{1-1/2^{k-5}+1/(m 2^{k-5})} = \frac{1}{\alpha}.$$

This contradicts the property  $L$ . Therefore,  $\chi \geq m$ . Finally, let us prove that  $\chi = m$ . Suppose that  $\chi > m$ . Remove from  $X$  paths with length  $2^{k-5}$  which connect vertices  $a, b$  (without the vertices  $a, b$ ) in such a way that  $m+1$  paths remain. Add to the remaining graph paths with length  $2^{k-5}$  from  $\mathcal{G}$  which connect  $a$  and vertices from  $N_{2^{k-6}}(a, b)$  (one path for each vertex) such that an intersection of any two of these paths equals  $\{a\}$  and an intersection of any of these paths with any path from  $X$  contains  $a$  and one vertex from  $N_{2^{k-6}}(a, b)$  only. Denote the final graph by  $\tilde{X}$ . Then  $\rho(\tilde{X}) = \frac{2^{k-4}(m+1)}{2(2^{k-5}-1)(m+1)+2} > 1/\alpha$ . This contradicts Property 2 in the definition of  $\tilde{\Omega}_n$ .

So,  $\chi = m$ . Let  $z \neq a$  be a vertex such that the predicate  $D_{2^{k-5}}^*(\cdot, z)$  is true for all vertices from  $N_{2^{k-6}}(a, b)$ . Then in  $\mathcal{G}$  there exist paths  $P_1, \dots, P_m$  with length  $2^{k-5}$  which connect a vertex  $z$  with vertices  $x^1, \dots, x^m$  respectively. Suppose that for some  $i \in \{1, \dots, m-1\}$   $P_{i+1} \subseteq P_1 \cup \dots \cup P_i$ . Set

$$P_{i+1} = (\{x^{i+1}, v_1, \dots, v_{2^{k-5}-1}, z\}, \{\{x^{i+1}, v_1\}, \{v_1, v_2\}, \dots, \{v_{2^{k-5}-1}, z\}\}).$$



Then for some  $j \in \{1, \dots, i\}$  the vertex  $v_1$  is in  $V(P_j)$ . Obviously,  $v_1 \neq x^j$  (otherwise, the predicate  $D_{2^{k-5}-1}(x^j, z)$  is true). Suppose that  $v_1 \approx x^j$  in  $\mathcal{G}$ . Then the predicate  $D_s(z, v_1)$  is true for some natural  $s < 2^{k-5} - 1$ . As  $v_1 \sim x^{i+1}$ , the predicate  $D_{s+1}(x^{i+1}, z)$  is true as well. This contradicts the truth of the predicate  $D_{2^{k-5}}^*(x^{i+1}, z)$ . Therefore, the vertex  $v_1$  is a common neighbor of the vertices  $x^{i+1}$  and  $x^j$ . This contradicts the truth of the predicate  $S(a, b)$ . So, for any  $i \in \{1, \dots, m-1\}$   $P_{i+1} \not\subseteq P_1 \cup \dots \cup P_i$ . Let us replace the graph  $X$  with its union with paths with length  $2^{k-5}$  from  $\mathcal{G}$  which connect  $a$  and vertices from  $N_{2^{k-6}}(a, b)$  (one path for each vertex) such that an intersection of any two of these paths equals  $\{a\}$  and an intersection of any of these paths with any path from  $X$  contains  $a$  and one vertex from  $N_{2^{k-6}}(a, b)$  only. Consider the sequence of graphs  $X_0 = X$ ,  $X_1 = X \cup P_1$ ,  $X_2 = X \cup P_1 \cup P_2$ ,  $\dots$ ,  $X_m = X \cup P_1 \cup \dots \cup P_m$ . Set  $Y := X_m$ . For any  $i \in \{0, \dots, m-1\}$ , the graph  $X_{i+1}$  is obtained from the graph  $X_i$  by adding  $n_i \leq 2^{k-5} - 1$  vertices and  $e_i \geq n_i + 1$  edges. Therefore,

$$1/\rho(Y) \leq \frac{2(2^{k-5} - 1)m + 2 + n_1 + \dots + n_m + 1}{2^{k-4}m + n_1 + \dots + n_m + m} \leq \alpha,$$

Equalities hold if and only if  $n_i = 2^{k-5} - 1$  and  $e_i = 2^{k-5}$  for all  $i \in \{0, \dots, m-1\}$ . Therefore, by the definition of the set  $\tilde{\Omega}_n$  these equalities hold and  $1/\rho(Y) = 1/\rho(X) = \alpha$ . As in  $\mathcal{G}$  there is no vertex  $z$  which follows the above properties, the graph  $\mathcal{G}$  does not contain a copy of  $Y$ , which contains the graph  $X$ .

As in Theorem 1, in what follows we exploit the notations  $X$  and  $Y$  for two obtained graphs with the density  $1/\alpha$  (obviously, the graph  $X$  and the pair  $(Y, X)$  are strictly balanced). Moreover, denote the obtained property of  $\mathcal{G}$  (existence of a copy of  $X$  such that any copy of  $Y$  does not contain it) by  $\tilde{L}$ . We proved that if  $\mathcal{G} \in \tilde{\Omega}_n$  and  $\mathcal{G}$  follows  $L$ , then  $\mathcal{G}$  follows  $\tilde{L}$  as well.

Finally, suppose that  $\mathcal{G} \in \tilde{\Omega}_n$  and  $\mathcal{G}$  follows  $\tilde{L}$ . Then, obviously,  $\mathcal{G}$  follows  $L$  as well.

By Lemma 1, there exists a partial limit  $\lim_{i \rightarrow \infty} \mathbb{P}(G(n_i, n_i^{-\alpha}) \models \tilde{L}) = c$ , which is not 0 or 1. Moreover, Equation (1) hold. Since  $1 - \frac{1}{2^{k-5}} + \frac{1}{2^{k-5}m} \rightarrow 1 - \frac{1}{2^{k-5}}$  as  $m \rightarrow \infty$ , the theorem is proved.

## 4.3 Proof of Theorem 3

We start the proof from the statement of the theorem of Ehrenfeucht in Section 4.3.1. This theorem is the main tool in proofs of zero-one laws. Then in Section 4.3.2 we define some supplementary constructions (cyclic extensions), after which in Section 4.3.3 we describe asymptotic properties of the random graph which imply the existence of a winning strategy of Duplicator. This strategy is described in Sections 4.3.4–4.3.8.

### 4.3.1 Ehrenfeucht game

In this section, we state a particular case of Ehrenfeucht theorem (see [4]), which holds for graphs. First, let us define Ehrenfeucht game  $\text{EHR}(G, H, i)$  on graphs  $G, H$  and  $i$  rounds (see, e.g., [7, 22]). Let  $V(G) = \{x_1, \dots, x_n\}$ ,  $V(H) = \{y_1, \dots, y_m\}$ . In the  $\nu$ -th round ( $1 \leq \nu \leq i$ ), Spoiler chooses a vertex in any graph (he chooses either  $x_{j_\nu} \in V(G)$  or  $y_{j'_\nu} \in V(H)$ ). Then Duplicator chooses any vertex in the other graph. If Spoiler chooses in the  $\mu$ -th round, say, the vertex  $x_{j_\mu} \in V(G)$ ,  $j_\mu = j_\nu$  ( $\nu < \mu$ ), then Duplicator must choose the vertex  $y_{j'_\mu} \in V(H)$ . If in this round Spoiler chooses, say, a vertex  $x_{j_\mu} \in V(G)$ ,  $j_\mu \notin \{j_1, \dots, j_{\mu-1}\}$ , then Duplicator must choose a vertex  $y_{j'_\mu} \in V(H)$  such that  $j'_\mu \notin \{j'_1, \dots, j'_{\mu-1}\}$ . If he can not do this, Spoiler wins. After the last round vertices  $x_{j_1}, \dots, x_{j_i} \in V(G)$  and  $y_{j'_1}, \dots, y_{j'_i} \in V(H)$  are chosen. If some of these vertices coincide, then leave out the copies and consider only distinct vertices:  $x_{h_1}, \dots, x_{h_l}; y_{h'_1}, \dots, y_{h'_l}$ ,  $l \leq i$ . Duplicator wins if

and only if the corresponding subgraphs are isomorphic up to the order of the vertices::

$$G|_{\{x_{h_1}, \dots, x_{h_l}\}} \cong H|_{\{y_{h'_1}, \dots, y_{h'_l}\}}.$$

**Theorem 8 ([4])** *For any graphs  $G, H$  and any  $i \in \mathbb{N}$ , Duplicator has a winning strategy in the game  $\text{EHR}(G, H, i)$  if and only if for any property  $L$  which is expressed by a first-order formula with the quantifier depth at most  $i$  either both graphs follow  $L$  or both graphs do not follow  $L$ .*

It can be easily shown that this theorem has the following corollary related to the zero-one laws (see, e.g., [22]).

**Theorem 9** *The random graphs  $G(n, p)$  obeys zero-one  $k$ -law if and only if*

$$\lim_{n, m \rightarrow \infty} \text{P}(\text{Duplicator has a winning strateg in } \text{EHR}(G(n, p(n)), G(m, p(m)), k)) = 1.$$

### 4.3.2 Constructions

Let  $m \geq 2$  be an arbitrary natural number. Consider a pair of graphs  $(G, H)$  such that  $G \supset H$ . We say that  $G$  is a *cyclic  $m$ -extension of  $H$* , if one of the following properties holds.

- The inequality  $m \geq 3$  holds. Moreover, there exists a vertex  $x_1$  of  $G$  such that

$$V(G) \setminus V(H) = \{y_1^1, \dots, y_{t_1}^1, y_1^2, \dots, y_{t_2}^2\},$$

$$E(G) \setminus E(H) = \{\{x_1, y_1^1\}, \{y_1^1, y_2^1\}, \dots, \{y_{t_1-1}^1, y_{t_1}^1\}, \{y_{t_1}^1, y_1^2\}, \{y_1^2, y_2^2\}, \dots, \{y_{t_2-1}^2, y_{t_2}^2\}, \{y_{t_2}^2, y_1^1\}\},$$

where  $t_1 + t_2 \leq m - 1$ ,  $t_1 \geq 0$ ,  $t_2 \geq 2$  (if  $t_1 = 0$ , then the vertex  $x_1$  is adjacent to vertices  $y_1^2, y_{t_2}^2$ ). In such a situation,  $G$  is the *first type* extension.

- The inequality  $m \geq 2$  holds. Moreover, there exist two distinct vertices  $x_1, x_2$  of  $G$  such that for some  $t \leq m - 1$

$$G = (V(H) \sqcup \{y_1, \dots, y_t\}, E(H) \sqcup \{\{x_1, y_1\}, \{y_1, y_2\}, \dots, \{y_{t-1}, y_t\}, \{y_t, x_2\}\}).$$

In such a situation,  $G$  is the *second type* extension.

Let  $H \subset G$  be two subgraphs in a graph  $\Gamma$ . The pair  $(G, H)$  is *cyclically  $m$ -maximal in  $\Gamma$* , if there are no cyclic  $m$ -extensions of  $G$  in  $\Gamma$  which are not cyclic  $m$ -extensions of  $H$ .

### 4.3.3 Properties which imply the existence of Duplicator's winning strategy

Let  $k > 3$ ,  $b$  be arbitrary natural numbers,  $\frac{a}{b}$  be an irreducible positive fraction,  $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$ ,  $p = n^{-\alpha}$ . Moreover, let  $a \in \{\max\{1, 2^{k-1} - b\}, \dots, 2^{k-1}\}$ .

Let us define a set of graphs  $\mathcal{S}$ . A graph  $G$  is in  $\mathcal{S}$  if and only if it follows three properties below.

- 1) In  $G$ , there are no strictly balanced subgraphs with at most  $2^{2k}b$  vertices and a density greater than  $1/\alpha$ .

- 2) Let  $\mathcal{H}$  be a set of  $\alpha$ -safe pairs  $(H_1, H_2)$  such that  $v(H_1) \leq 2^{2k}b + k2^k$ . Let  $\mathcal{K}$  be a set of pairs  $(K_1, K_2)$  such that  $v(K_1) \leq 2^k$ ,  $v(K_2) \leq 2$  and  $f_\alpha(K_1, K_2) < 0$ . Then for any pair  $(H_1, H_2) \in \mathcal{H}$ ,  $V(H_2) = \{v_1, \dots, v_h\}$ , and for any subgraph  $G_2 \subset G$ ,  $V(G_2) = \{x_1, \dots, x_h\}$ , in  $G$  there exists a strict  $(H_1, (v_1, \dots, v_h))$ -extension  $G_1$  of the ordered tuple  $(x_1, \dots, x_h)$  such that the pair  $(G_1, G_2)$  is  $(K_1, K_2)$ -maximal in  $G$  for any pair  $(K_1, K_2) \in \mathcal{K}$ .
- 3) Let  $\mathcal{H}$  be a set of pairs  $(H_1, H_2)$  such that  $v(H_1) \leq 2^k$ ,  $v(H_2) \leq 2$  and  $f_\alpha(H_1, H_2) < 0$ . Then, for any strictly balanced graph  $H$  with at most  $2^{2k}b$  vertices and  $\rho(H) < 1/\alpha$ , in  $G$  there is a copy of  $H$  which is  $(H_1, H_2)$ -maximal in  $G$  for any  $(H_1, H_2) \in \mathcal{H}$ .

By Theorem 4, Theorem 7 and Corollary 2,  $P(G(n, p) \in \mathcal{S}) \rightarrow 1$  as  $n \rightarrow \infty$ . Therefore, by Theorem 9, the statement of Theorem 3 follows from the existence of a winning strategy of Duplicator in  $\text{EHR}(G, H, k)$  for all pairs  $(G, H)$  such that  $G, H \in \mathcal{S}$ .

#### 4.3.4 Winning strategy of Duplicator

Let  $G, H \in \mathcal{S}$ . Let  $X_r, Y_r$  be chosen in the  $r$ -th round graphs by Spoiler and Duplicator respectively. So, the sets  $\{X_r, Y_r\}$  and  $\{G, H\}$  coincide for all  $r \in \{1, \dots, k\}$ . We denote vertices which are chosen in the first  $r$  rounds in  $X_r$  and  $Y_r$  by  $x_r^1, \dots, x_r^r$  and  $y_r^1, \dots, y_r^r$  respectively. Let us describe Duplicator's strategy by induction. The strategy is divided into two parts. We denote the first and second strategy by **S** and **SF** respectively. In the first round, Duplicator always use the strategy **S** and follows this strategy until a round such that chosen subgraphs allow to exploit the strategy **SF**, which was introduced in [18] (we do not describe this strategy in the presented paper, because its detailed description can be found in [19], Section 4.8).

Before describe the strategies, we introduce one more important notion. Let  $r$  rounds are finished,  $r \in \{1, \dots, k\}$ . Let  $l \in \{1, \dots, r\}$  and graphs  $\tilde{X}_r^1, \dots, \tilde{X}_r^l \subset X_r$ ,  $\tilde{Y}_r^1, \dots, \tilde{Y}_r^l \subset Y_r$  which do not have common vertices satisfy the following properties.

- I The verices  $x_r^1, \dots, x_r^r$  are elements of the set  $V(\tilde{X}_r^1 \cup \dots \cup \tilde{X}_r^l)$ , the vertices  $y_r^1, \dots, y_r^r$  are elements of the set  $V(\tilde{Y}_r^1 \cup \dots \cup \tilde{Y}_r^l)$ .
- II For any distinct  $j_1, j_2 \in \{1, \dots, l\}$ , the inequalities  $d_{X_r}(\tilde{X}_r^{j_1}, \tilde{X}_r^{j_2}) > 2^{k-r}$ ,  $d_{Y_r}(\tilde{Y}_r^{j_1}, \tilde{Y}_r^{j_2}) > 2^{k-r}$  hold.
- III For any  $j \in \{1, \dots, l\}$ , in the graph  $X_r$  (in the graph  $Y_r$ ) there is no cyclic  $2^{k-r}$ -extension of the graph  $\tilde{X}_r^j$  (the graph  $\tilde{Y}_r^j$ ).
- IV Cardinalities of the sets  $V(\tilde{X}_r^1 \cup \dots \cup \tilde{X}_r^l)$ ,  $V(\tilde{Y}_r^1 \cup \dots \cup \tilde{Y}_r^l)$  are at most  $2^{2k}b + 2^{k-1}r$ .
- V The graphs  $\tilde{X}_r^j$  and  $\tilde{Y}_r^j$  are isomorphic for any  $j \in \{1, \dots, l\}$  and there exists a corresponding isomorphism (one for all these pairs of graphs) which maps the vertices  $x_r^i$  to the vertices  $y_r^i$ ,  $i \in \{1, \dots, r\}$ .

Two ordered tuples of graphs  $\tilde{X}_r^1, \dots, \tilde{X}_r^l$  and  $\tilde{Y}_r^1, \dots, \tilde{Y}_r^l$  which follow the above properties we call  $(k, r, l)$ -regular equivalent in  $(X_r, Y_r)$ . Moreover, we denote an isomorphism from Property V by  $\varphi(k, r, l)$  (generally speaking, such an isomorphism is not unique, therefore, we consider an arbitrary isomorphism from Property V).

Note that  $(k, 1, 1)$ -regular equivalence of  $\tilde{X}_1^1$  and  $\tilde{Y}_1^1$  is defined by Properties I, III, IV and V. Moreover,  $(k, k, l)$ -regular equivalence of ordered tuples  $\tilde{X}_k^1, \dots, \tilde{X}_k^l$  and  $\tilde{Y}_k^1, \dots, \tilde{Y}_k^l$  is defined by

Properties I, II, IV and V.

Two graphs  $\tilde{X}_r^1$  and  $\tilde{Y}_r^1$  are called  $(k, r)$ -equivalent in  $(X_r, Y_r)$ , if for  $l = 1$  Properties I, IV and V hold and in the graph  $X_r$  (the graph  $Y_r$ ) there is no cyclic  $2^{k-r} - 1$ -extension of the graph  $\tilde{X}_r^1$  (the graph  $\tilde{Y}_r^1$ ), there is no second type cyclic  $2^{k-r}$ -extension of the graph  $X_r|_{\{x_r^1, \dots, x_r^r\}}$  (the graph  $Y_r|_{\{y_r^1, \dots, y_r^r\}}$ ) and there exists at most one cyclic  $2^{k-r}$ -extension of the graph  $\tilde{X}_r^1$  (the graph  $\tilde{Y}_r^1$ ).

The main idea of Duplicator's strategy is the following. Duplicator should play in such a way that for some  $r \in \{1, \dots, k-1\}$  and  $l \in \{1, \dots, r\}$  in the graphs  $X_r, Y_r$   $(k, r, l)$ -regular equivalent ordered tuples of subgraphs in  $(X_r, Y_r)$  are constructed. In the first round, Duplicator must use the strategy  $S_1$  which is described in the next section. After the  $r$ -th round,  $r \in \{1, \dots, k-3\}$ , if  $(k, r, l)$ -regular equivalent ordered tuples are not constructed, then, as we show, Duplicator either can find  $(k, r)$ -equivalent graphs (and then, in the  $r+1$ -th round, he must use the strategy  $S_{r+1}$ , which is described in Section 4.3.6) or he must use the strategy  $S_{r+1}^1$ , which is described in Section 4.3.7. After the strategy  $S_{r+1}^1$ , Duplicator never turns back to the strategy  $S_{r+j}$ ,  $j \geq 2$ . Strategy SF is described in [19] (Section 4.8) and is used by Duplicator in the  $r+1$ -th round,  $r \geq 2$ , if and only if after the  $r$ -th round for some  $l \in \{1, \dots, r\}$   $(k, r, l)$ -regular equivalent ordered tuples of graphs in  $(X_r, Y_r)$  are constructed. In [19], it is proved that Duplicator wins, when he uses the strategy SF.

#### 4.3.5 Strategy $S_1$

Consider the first round and two possibilities to choose the first vertex by Spoiler.

Let in  $X_1$  there is no cyclic  $2^{k-1}$ -extension of  $(\{x_1^1\}, \emptyset)$ . Then Duplicator chooses a vertex  $y_1^1 \in V(Y_1)$  which satisfies the following property (such a vertex exists because  $Y_1 \in \mathcal{S}$  and, therefore,  $Y_1$  satisfies 3)). There are no cyclic  $2^{k-1}$ -extensions of  $(\{y_1^1\}, \emptyset)$  in  $Y_1$ . Set  $\tilde{X}_1^1 = (\{x_1^1\}, \emptyset)$ ,  $\tilde{Y}_1^1 = (\{y_1^1\}, \emptyset)$ . Property III of  $(k, 1, 1)$ -regular equivalence of the graphs  $\tilde{X}_1^1$  and  $\tilde{Y}_1^1$  in  $(X_1, Y_1)$  is already proved. Obviously, Properties I, IV and V hold as well. In this case, in the second round Duplicator exploits the strategy SF.

Let in  $X_1$  there exists at least one cyclic  $2^{k-1}$ -extension  $\tilde{X}_1^1$  of  $(\{x_1^1\}, \emptyset)$ . Let us prove that there exists a sequence of graphs  $G_1, G_2, \dots, G_s$  such that

- a) for any  $i \in \{1, \dots, s-1\}$ , the graph  $G_{i+1}$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_i$  in  $X_1$ ,  $G_1$  is a cyclic  $2^{k-1}$ -extension of the graph  $(\{x_1^1\}, \emptyset)$ ,
- b) either  $X_1|_{V(G_s)} = G_s$ , or  $\rho(X_1|_{V(G_s)}) < 1/\alpha$ ,
- c) there are no cyclic  $2^{k-1}$ -extensions of  $G_s$  in  $X_1$ ,
- d) if for some  $i \in \{1, \dots, s-1\}$  the graph  $G_{i+1}$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_i$ , but it is not a cyclic  $2^{k-1} - 1$ -extension of the graph  $G_i$ , then there exists  $\mu \in \{1, \dots, s-1\}$  such that the graph  $G_{\mu+1}$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_\mu$ , but it is not a cyclic  $2^{k-1} - 1$ -extension of the graph  $G_\mu$ , while in the graph  $X_1 \setminus (G_{\mu+1} \setminus G_\mu)$  there is no cyclic  $2^{k-1}$ -extensions of the graph  $G_\mu$ .

Let us prove the existence of such a sequence.

Obviously, there exists a sequence  $G_1 \subset G_2 \dots \subset G_i$  with the following properties. First,  $G_1$  is a cyclic  $2^{k-1}$ -extension of the graph  $(\{x_1^1\}, \emptyset)$ ,  $G_j$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_{j-1}$  for any

$j \in \{2, \dots, i\}$ . Second,  $j = i$  is the first number (if such a number exists) such that  $G_j$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_{j-1}$ , but it is not a cyclic  $2^{k-1} - 1$ -extension of the graph  $G_{j-1}$  (here,  $G_0 = (\{x_1^1\}, \emptyset)$ ). If such a number does not exist, then there are no cyclic  $2^{k-1}$ -extensions of  $G_i$  in  $X_1$  (obviously,  $i$  exists and  $i \leq 2^{k-1}b + 1$ , because a density of  $G_i$  is greater than  $1/\alpha$ , if  $i = 2^{k-1}b + 2$ , this contradicts Property 1)). In the last situation, the sequence  $G_1, \dots, G_s$  ( $s = i$ ), which satisfies Properties a), c) and d), is already built. Nevertheless, if  $G_i$  is not the “last” extension, then consider an arbitrary cyclic  $2^{k-1}$ -extension  $\hat{G}_i$  of  $G_{i-1}$  in  $X_1 \setminus (G_i \setminus G_{i-1})$  (if such an extension exists). Let us add cyclic  $2^{k-1}$ -extensions  $\hat{G}_{i+1}, \hat{G}_{i+2}, \dots$  of previously constructed graphs one by one in a similar way until there are no cyclic  $2^{k-1}$ -extensions of the graph  $\hat{G}_s$  in  $X_1 \setminus (G_i \setminus G_{i-1})$ . Obviously, the graph  $\hat{G}_s \cup G_i$  is a cyclic  $2^{k-1}$ -extension of the graph  $\hat{G}_s$ , but it is not its cyclic  $2^{k-1} - 1$ -extension. Moreover, there are no cyclic  $2^{k-1}$ -extensions of  $\hat{G}_s$  in  $X_1 \setminus ((\hat{G}_s \cup G_i) \setminus \hat{G}_s)$ . So, the first  $\hat{s} + 1$  graphs of the sequence are constructed:  $G_1, \dots, G_{i-1}, \hat{G}_i, \dots, \hat{G}_s, \hat{G}_s \cup G_i$ . Let us add cyclic  $2^{k-1}$ -extensions to the graph  $\hat{G}_s \cup G_i$  (each next graph is an extension of the previous one) until there are no cyclic  $2^{k-1}$ -extensions of the final graph in  $X_1$ . Obviously, we get the sequence of graphs (we denote it by  $G_1, G_2, \dots, G_s$ ), which follows Properties a), c) and d) (in addition, the inequality  $s \leq 2^{k-1}b + 1$  holds, because a density of the graph  $G_s$  is greater than  $1/\alpha$ , if  $s = 2^{k-1}b + 2$ , this contradicts Property 1)).

Suppose that  $e(X_1|_{V(G_s)}) > e(G_s)$ . Moreover, let  $e(X_1|_{V(G_s)}) - e(G_s) \geq 2$ . Since  $s \leq 2^{k-1}b + 1$ , by Property 1) the inequalities  $\rho^{\max}(X_1|_{V(G_s)}) \leq 1/\alpha < 1 + \frac{1}{2^{k-1}-1}$  hold. Then

$$1 + \frac{1}{2^{k-1}-1} > \rho^{\max}(X_1|_{V(G_s)}) \geq \frac{2^{k-1} + 2}{2^{k-1}} = 1 + \frac{1}{2^{k-2}}.$$

This contradicts the inequality  $k > 3$ . So,  $e(X_1|_{V(G_s)}) - e(G_s) = 1$ . For any  $i \in \{1, \dots, s\}$ , set  $e(G_i) - e(G_{i-1}) = e_i \leq 2^{k-1}$ , where  $G_0 = (\{x_1^1\}, \emptyset)$ . Then

$$1/\rho(X_1|_{V(G_s)}) = \frac{e_1 + \dots + e_s - s + 1}{e_1 + \dots + e_s + 1} = 1 - \frac{1}{2^{k-1} + \frac{(e_1 - 2^{k-1}) + \dots + (e_s - 2^{k-1}) + 1}{s}}.$$

Therefore, either  $1/\rho(X_1|_{V(G_s)}) = 1 - \frac{1}{2^{k-1} + \frac{1}{s}}$ , or  $1/\rho(X_1|_{V(G_s)}) \leq 1 - \frac{1}{2^{k-1}} < \alpha$ , where the last inequality holds, if at least one of  $e_i$ ,  $i \in \{1, \dots, s\}$ , is at most  $2^{k-1} - 1$ . In the last case, we arrive at a contradiction with Property 1) of the graph  $X_1$ , because  $s \leq 2^{k-1}b + 1$ . So,  $1/\rho(X_1|_{V(G_s)}) = 1 - \frac{1}{2^{k-1} + \frac{1}{s}}$  and  $e_1 = \dots = e_s = 2^{k-1}$ . If  $1/\rho(X_1|_{V(G_s)}) > \alpha$ , then Property b) holds. Moreover, the inequality  $1/\rho(X_1|_{V(G_s)}) < \alpha$  contradicts Property 1) of the graph  $X_1$ . Therefore,  $1 - \frac{1}{2^{k-1} + a/b} = 1 - \frac{1}{2^{k-1} + 1/s}$ . As  $a/b$  — the irreducible fraction,  $a = 1$ ,  $b = s$ . The last equalities hold only if  $2^{k-1} - b \leq 1$ . So,  $s \geq 7$ . Denote vertices of the additional edge, which exists according to our proposition, by  $u, v$ . Let  $u, v \in V(G_{s-1})$ . Moreover, let  $u \in V(G_{j_1+1}) \setminus V(G_{j_1})$ ,  $v \in V(G_{j_2+1}) \setminus V(G_{j_2})$ , where  $0 \leq j_1 \leq j_2 \leq s - 2$ ,  $G_0 = (\emptyset, \emptyset)$ . Obviously, if the set  $V(G_{j_2+1}) \setminus V(G_{j_2})$  contains more than one vertex, then there exist graphs  $\tilde{G}_{j_2+1}, \dots, \tilde{G}_{s+1}$  such that for any  $j \in \{j_2, \dots, s\}$  the graph  $\tilde{G}_{j+1}$  is a cyclic  $2^{k-1}$ -extension of the graph  $\tilde{G}_j$ , where  $\tilde{G}_{j_2} = G_{j_2}$  and for any  $j \in \{j_2 + 2, \dots, s + 1\}$  the equality  $\tilde{G}_j = X_1|_{V(G_{j-1})}$  holds. If  $v(G_{j_2+1}, G_{j_2}) = 1$ , then

$$1 + \frac{1}{2^{k-1}-1} > \rho^{\max}(X_1|_{V(G_{j_2+1})}) \geq \frac{2^{k-1} + 3}{2^{k-1} + 1} = 1 + \frac{1}{2^{k-2} + 1/2}.$$

This contradicts the inequality  $k > 3$ . Obviously, the sequence  $G_1, \dots, G_{i-1}, \tilde{G}_i, \dots, \tilde{G}_{s+1}$  follows Properties a)–d) (here,  $\tilde{G}_{s+1}$  is the cyclic  $2^{k-1}$ -extension of the graph  $\tilde{G}_s$  from Property d)). Finally,

let at least one of the vertices  $u, v$  (e.g.,  $v$ ) is from the set  $V(G_s) \setminus V(G_{s-1})$ . If the graph  $G_s \setminus (G_{s-1} \setminus G_{s-2})$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_{s-2}$  and  $u \notin V(G_{s-1}) \setminus V(G_{s-2})$ , then set  $G_{s-1} := G_s \setminus (G_{s-1} \setminus G_{s-2})$ . So, we get the above situation, which is already considered. If either the graph  $G_s \setminus (G_{s-1} \setminus G_{s-2})$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_{s-2}$  and  $u \in V(G_{s-1}) \setminus V(G_{s-2})$ , or  $G_s \setminus (G_{s-1} \setminus G_{s-2})$  is not a cyclic  $2^{k-1}$ -extension of the graph  $G_{s-2}$ , then there exist graphs  $\tilde{G}_s, \tilde{G}_{s+1}$  such that the graph  $\tilde{G}_s$  is a cyclic  $2^{k-1}$ -extension of the graph  $G_{s-1}$ , the graph  $\tilde{G}_{s+1}$  is a cyclic  $2^{k-1}$ -extension of the graph  $\tilde{G}_s$ ,  $\tilde{G}_{s+1} = X_1|_{V(G_s)}$ , and there are no cyclic  $2^{k-1}$ -extension of the graph  $G_{s-1}$  in  $X_1 \setminus (\tilde{G}_s \setminus G_{s-1})$ . Therefore, the sequence  $G_1, \dots, G_{s-1}, \tilde{G}_s, \tilde{G}_{s+1}$  follows Properties a)–d).

So, let  $G_1, G_2, \dots, G_s$  be a sequence which follows Properties a)–d). Let us prove that the graph  $X_1|_{V(G_s)}$  is strictly balanced. Let  $\tilde{G}$  be an arbitrary proper subgraph in  $X_1|_{V(G_s)}$ . Denote  $\tilde{G}_1 = X_1|_{V(G_1)} \cap \tilde{G}$ . If  $\tilde{G}_1 \neq X_1|_{V(G_1)}$ , then  $v(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G}) \leq 2^{k-1} - 1$ ,  $e(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G}) \geq v(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G}) + 1$ . Therefore, a density of the graph  $\tilde{G} \cup X_1|_{V(G_1)}$  is at least

$$\frac{e(\tilde{G}) + v(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G}) + 1}{v(\tilde{G}) + v(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G})} > \min \left\{ \frac{e(\tilde{G})}{v(\tilde{G})}, 1 + \frac{1}{v(\tilde{G} \cup X_1|_{V(G_1)}, \tilde{G})} \right\} = \rho(\tilde{G}).$$

In the same way, it can be proved that  $\rho(X_1|_{V(G_s)}) \geq \rho(\tilde{G} \cup X_1|_{V(G_{s-1})}) \geq \dots \geq \rho(\tilde{G} \cup X_1|_{V(G_1)}) \geq \rho(\tilde{G})$ , where at least one of the inequalities is strict, because  $\tilde{G}$  is a proper subgraph in  $X_1|_{V(G_s)}$ . Therefore, the graph  $X_1|_{V(G_s)}$  is strictly balanced.

If  $\rho(X_1|_{V(G_s)}) < 1/\alpha$ , then set  $\tilde{X}_1^1 = X_1|_{V(G_s)}$ . By the definition of the graph  $Y_1$ , it has a subgraph  $\tilde{Y}_1^1$  isomorphic to  $\tilde{X}_1^1$  such that the following property holds. The graph  $\tilde{Y}_1^1$  is  $(K, T)$ -maximal for any pair  $(K, T)$  such that  $v(K) \leq 2^k$ ,  $v(T) \leq 2$  and  $f_\alpha(K, T) < 0$ . Let  $\varphi : \tilde{X}_1^1 \rightarrow \tilde{Y}_1^1$  be an isomorphism. Then Duplicator chooses the vertex  $y_1^1 := \varphi(x_1^1)$ . By the construction of the graphs  $\tilde{X}_1^1$  and  $\tilde{Y}_1^1$ , they do not have cyclic  $2^{k-1}$ -extensions in  $X_1$  and  $Y_1$  respectively. Therefore, the graphs  $\tilde{X}_1^1$  and  $\tilde{Y}_1^1$  are  $(k, 1, 1)$ -regular equivalent in  $(X_1, Y_1)$ . In the second round Duplicator exploits the strategy SF.

Let  $\rho(X_1|_{V(G_s)}) = 1/\alpha$ . Then  $\rho(G_s) = 1/\alpha$  as well. Set  $G_0 = (\{x_1^1\}, \emptyset)$ . For any  $i \in \{1, \dots, s\}$ , denote  $e_i = e(G_i, G_{i-1})$ . Then

$$1 + \frac{1}{2^{k-1} + a/b - 1} = \frac{e_1 + \dots + e_s}{e_1 + \dots + e_s - s + 1} = 1 + \frac{1}{\frac{e_1 + \dots + e_s}{s-1} - 1}.$$

Since  $a/b$  is the irreducible fraction,  $s \geq b+1$ . Obviously, the inequality  $a \geq \max\{1, 2^{k-1} - b\}$  implies the existence of  $\mu \in \{0, \dots, s-1\}$  such that  $G_{\mu+1}$  is not a cyclic  $2^{k-1} - 1$ -extension of the graph  $G_\mu$ . Indeed, otherwise

$$\begin{aligned} \rho(G_s) &\geq \frac{(2^{k-1} - 1)s}{(2^{k-1} - 2)s + 1} = 1 + \frac{1}{2^{k-1} - 2 + \frac{2^{k-1}-1}{s-1}} \geq 1 + \frac{1}{2^{k-1} - 2 + \frac{2^{k-1}-1}{b}} = \\ &= 1 + \frac{1}{2^{k-1} + \frac{2^{k-1}-2b-1}{b}} > 1/\alpha. \end{aligned}$$

Since  $G_s$  is strictly balanced,  $\rho^{\max}(G_\mu) < 1/\alpha$ . As  $Y_1 \in \mathcal{S}$ , in  $Y_1$  there exists a subgraph  $\tilde{Y}_1^1$  isomorphic to  $\tilde{X}_1^1 := G_\mu$  such that the following property holds. The graph  $\tilde{Y}_1^1$  is  $(K, T)$ -maximal for any pair  $(K, T)$  such that  $v(K) \leq 2^k$ ,  $v(T) \leq 2$  and  $f_\alpha(K, T) < 0$ . Let  $\varphi : \tilde{X}_1^1 \rightarrow \tilde{Y}_1^1$  be an isomorphism. Then Duplicator chooses the vertex  $y_1^1 := \varphi(x_1^1)$ . By the construction of the graphs  $\tilde{X}_1^1$  and  $\tilde{Y}_1^1$ , they are  $(k, 1)$ -equivalent in  $(X_1, Y_1)$ . Therefore, in the second round Duplicator exploits the strategy  $\mathcal{S}_2$ .

### 4.3.6 Strategy $S_{r+1}$

Let after the  $r$ -th round,  $r \in \{1, \dots, k-2\}$ , there exist graphs  $\tilde{X}_r^1, \tilde{Y}_r^1$  which are  $(k, r)$ -equivalent in  $(X_r, Y_r)$ . Let  $\varphi : \tilde{X}_r^1 \rightarrow \tilde{Y}_r^1$  be an automorphism.

In the  $r+1$ -th round, Spoiler chooses a vertex  $x_{r+1}^{r+1}$ . If  $X_{r+1} = X_r$ , then set  $\tilde{X}_{r+1}^1 = \tilde{X}_r^1, \tilde{Y}_{r+1}^1 = \tilde{Y}_r^1$ . Otherwise, set  $\tilde{X}_{r+1}^1 = \tilde{Y}_r^1, \tilde{Y}_{r+1}^1 = \tilde{X}_r^1$ .

Let  $x_{r+1}^{r+1} \in V(\tilde{X}_{r+1}^1)$ . Duplicator chooses the vertex  $y_{r+1}^{r+1} = \varphi(x_{r+1}^{r+1})$ , if  $X_{r+1} = X_r$ , and the vertex  $y_{r+1}^{r+1} = \varphi^{-1}(x_{r+1}^{r+1})$ , if  $X_{r+1} = Y_r$ . As in  $X_r, Y_r$  there are no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{X}_r^1, \tilde{Y}_r^1$  respectively (by the definition of the  $(k, r)$ -equivalence), the graphs  $\tilde{X}_{r+1}^1, \tilde{Y}_{r+1}^1$  are  $(k, r+1, 1)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Therefore, in the  $r+2$ -th round Duplicator exploits the strategy SF.

Let  $x_{r+1}^{r+1} \notin V(\tilde{X}_{r+1}^1)$ . Consider two cases:  $r < k-2$  and  $r = k-2$ .

Let  $r < k-2$ . If  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, x_{r+1}^{r+1}) > 2^{k-r-1}$  and in  $X_{r+1}$  there are no cyclic  $2^{k-r-1}$ -extensions of the graph  $(\{x_{r+1}^{r+1}\}, \emptyset)$ , then set  $\tilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \emptyset)$ . By Property 2) of the graph  $Y_{r+1}$ , it has a vertex  $y_{r+1}^{r+1}$  such that  $d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, y_{r+1}^{r+1}) = 2^{k-r-1} + 1$  and there are no cyclic  $2^{k-r-1}$ -extensions of  $(\{y_{r+1}^{r+1}\}, \emptyset)$  in  $Y_{r+1}$ . Set  $\tilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\}, \emptyset)$ . If there is exactly one cyclic  $2^{k-r-1}$ -extension of  $(\{x_{r+1}^{r+1}\}, \emptyset)$ , then we denote it by  $\tilde{X}_{r+1}^2$ . Let  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) > 2^{k-r-1}$ . By Property 2) of the graph  $\tilde{Y}_{r+1}^1$ , it has a vertex  $y_{r+1}^{r+1}$  and a subgraph  $\tilde{Y}_{r+1}^2$  such that  $d_{Y_2}(\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2) = 2^{k-r-1} + 1$ , pairs  $(\tilde{Y}_{r+1}^2, (\{y_{r+1}^{r+1}\}, \emptyset))$  and  $(\tilde{X}_{r+1}^2, (\{x_{r+1}^{r+1}\}, \emptyset))$  are isomorphic, and there are no cyclic  $2^{k-r-1}$ -extensions of  $\tilde{Y}_{r+1}^2$  in  $Y_{r+1}$ . The property of  $(k, r)$ -equivalence of the graphs  $\tilde{X}_r^1, \tilde{Y}_r^1$  in  $(X_r, Y_r)$  implies non-existence of cyclic  $2^{k-r-1}$ -extensions of  $\tilde{X}_r^1$  and  $\tilde{Y}_r^1$  in  $X_r$  and  $Y_r$  respectively. Obviously, in all the considered cases the ordered tuples  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  and  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  are  $(k, r+1, 2)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Thus, in the  $r+2$ -th round Duplicator exploits the strategy SF. Let  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) \leq 2^{k-r-1}$ . The property of  $(k, r)$ -equivalence of the graphs  $\tilde{X}_r^1, \tilde{Y}_r^1$  in  $(X_r, Y_r)$  implies  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) = 2^{k-r-1}$  and non-existence of cyclic  $2^{k-r-1} - 1$ -extensions of  $(\{x_{r+1}^{r+1}\}, \emptyset)$  in  $X_{r+1}$ . In this case, by Property 2) of the graph  $Y_{r+1}$  Duplicator is able to choose a vertex  $y_{r+1}^{r+1}$  such that the following property holds. There exists an isomorphism  $L_X \cup \tilde{X}_{r+1}^1 \rightarrow L_Y \cup \tilde{Y}_{r+1}^1$  which maps the vertices  $x_{r+1}^1, \dots, x_{r+1}^{r+1}$  to the vertices  $y_{r+1}^1, \dots, y_{r+1}^{r+1}$  respectively, where  $L_X$  is a minimal path in  $X_{r+1}$  which connects  $x_{r+1}^{r+1}$  and  $\tilde{X}_{r+1}^1$ ,  $L_Y$  is a minimal path in  $Y_{r+1}$  which connects  $y_{r+1}^{r+1}$  and  $\tilde{Y}_{r+1}^1$ , and the pair  $(L_Y \cup \tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^1)$  is cyclically  $2^{k-r-1}$ -maximal in  $Y_{r+1}$ . Set  $\tilde{X}_{r+1}^1 := \tilde{X}_{r+1}^1 \cup L_X, \tilde{Y}_{r+1}^1 := \tilde{Y}_{r+1}^1 \cup L_Y$ . Next, Duplicator exploits the strategy  $S_{r+2}^1$ . Finally, let us prove the the graph  $(\{x_{r+1}^{r+1}\}, \emptyset)$  has at most one cyclic  $2^{k-r-1}$ -extension. Indeed, if two such extensions  $A$  and  $\tilde{A}$  exist, then

$$1/\rho(A \cup \tilde{A}) \leq \frac{2^{k-r-1} + 2^{k-r-1} - 1}{2^{k-r-1} + 2^{k-r-1}} = 1 - \frac{1}{2^{k-r}} < \alpha.$$

This contradicts Property 1), since  $v(A \cup \tilde{A}) \leq 2^{k-r} - 1$ .

Let  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, x_{r+1}^{r+1}) \leq 2^{k-r-1}$ . Consider a minimal path  $L_X$  in  $X_{r+1}$  which connects  $x_{r+1}^{r+1}$  and  $\tilde{X}_{r+1}^1$ . By Property 2) of the graph  $Y_{r+1}$ , there exists a vertex  $y_{r+1}^{r+1}$  such that  $d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, y_{r+1}^{r+1}) = d_{X_{r+1}}(\tilde{X}_{r+1}^1, x_{r+1}^{r+1})$ , there exists an isomorphism  $L_X \cup \tilde{X}_{r+1}^1 \rightarrow L_Y \cup \tilde{Y}_{r+1}^1$  which maps the vertices  $x_{r+1}^1, \dots, x_{r+1}^{r+1}$  to the vertices  $y_{r+1}^1, \dots, y_{r+1}^{r+1}$  respectively, and the pair  $(L_Y \cup \tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^1)$  is cyclically  $2^{k-r-1}$ -maximal, where  $L_Y$  is a minimal path which connects  $y_{r+1}^{r+1}$  and  $\tilde{Y}_{r+1}^1$  in  $Y_{r+1}$ . Obviously, there are no cyclic  $2^{k-r-1}$ -extensions of the graph  $\tilde{Y}_{r+1}^1$  in  $Y_{r+1}$ . Set  $\tilde{Y}_{r+1}^1 := \tilde{Y}_{r+1}^1 \cup L_Y$ . If there are no cyclic  $2^{k-r-1}$ -extensions of  $L_X \cup \tilde{X}_{r+1}^1$  in  $X_{r+1}$ , then set  $\tilde{X}_{r+1}^1 := L_X \cup \tilde{X}_{r+1}^1$ . Obviously, the graphs  $\tilde{X}_{r+1}^1$  and  $\tilde{Y}_{r+1}^1$  are  $(k, r+1, 1)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Therefore, in the next round Duplicator exploits the strategy SF. If there is a cyclic  $2^{k-r-1}$ -extension of  $L_X \cup \tilde{X}_{r+1}^1$  in  $X_{r+1}$ , then

$d_{X_{r+1}}(x_{r+1}^{r+1}, \tilde{X}_{r+1}^1) = 2^{k-r-1}$  and there are no cyclic  $2^{k-r-1} - 1$ -extensions of  $L_X \cup \tilde{X}_{r+1}^1$  in  $X_{r+1}$ . In this case, the path  $L_X$  could be chosen from a set with at most two paths. If there is one such path, then either a cyclic  $2^{k-r-1}$ -extension of the graph  $L_X \cup \tilde{X}_{r+1}^1$  is the first type extension, or one of the terminal vertices of  $L_X$  does not coincide with each of the vertices  $x_{r+1}^1, \dots, x_{r+1}^{r+1}$ . If there are two paths, then consider two cases. First, if a cyclic  $2^{k-r}$ -extension of the graph  $\tilde{X}_{r+1}^1$  is the first type extension, then we choose an arbitrary path  $L_X$  from these two paths. Second, if a  $2^{k-r}$ -extension of the graph  $\tilde{X}_{r+1}^1$  is the second type extension, then  $(k, r)$ -equivalence of the graphs  $\tilde{X}_{r+1}^1, \tilde{Y}_{r+1}^1$  in  $(X_{r+1}, Y_{r+1})$  imply that at least one path does not contain vertices  $x_{r+1}^1, \dots, x_{r+1}^r$ . In this case,  $L_X$  is such a path. Obviously, the graphs  $\tilde{X}_{r+1}^1 := L_X \cup \tilde{X}_{r+1}^1, \tilde{Y}_{r+1}^1$  are  $(k, r+1)$ -equivalent in  $(X_{r+1}, Y_{r+1})$ . In the next round, Duplicator exploits the strategy  $S_{r+2}$ .

Finally, let  $r = k - 2$ . If  $d_{X_{k-1}}(\tilde{X}_{k-1}^1, x_{k-1}^{k-1}) > 2$ , then set  $\tilde{X}_{k-1}^2 = (\{x_{k-1}^{k-1}\}, \emptyset)$ . By Property 2) of the graph  $Y_{k-1}$ , it contains a vertex  $y_{k-1}^{k-1}$  such that  $d_{Y_{k-1}}(\tilde{Y}_{k-1}^1, y_{k-1}^{k-1}) = 3$ . Set  $\tilde{Y}_{k-1}^2 = (\{y_{k-1}^{k-1}\}, \emptyset)$ . Since the graphs  $\tilde{X}_{k-2}^1, \tilde{Y}_{k-2}^1$  are  $(k, k-2)$ -equivalent in  $(X_{k-2}, Y_{k-2})$ , they do not have cyclic 2-extensions in  $X_{k-2}$  and  $Y_{k-2}$  respectively. Thus, the ordered tuples  $\tilde{X}_{k-1}^1, \tilde{X}_{k-1}^2$  and  $\tilde{Y}_{k-1}^1, \tilde{Y}_{k-1}^2$  are  $(k, k-1, 2)$ -regular equivalent in  $(X_{k-1}, Y_{k-1})$ . Therefore, in the  $k$ -th round Duplicator exploits the strategy SF.

If  $d_{X_{k-1}}(\tilde{X}_{k-1}^1, x_{k-1}^{k-1}) \leq 2$ , then consider a minimal path  $L_X$  in  $X_{k-1}$ , which connects  $x_{k-1}^{k-1}$  and  $\tilde{X}_{k-1}^1$ . Moreover, let this path connect  $x_{k-1}^{k-1}$  and one of the vertices  $x_{k-1}^1, \dots, x_{k-1}^{k-2}$ , if such a path with the minimal length exists. By Property 2) of the graph  $Y_{k-1}$ , it contains a vertex  $y_{k-1}^{k-1}$  such that  $d_{Y_{k-1}}(\tilde{Y}_{k-1}^1, y_{k-1}^{k-1}) = d_{X_{k-1}}(\tilde{X}_{k-1}^1, x_{k-1}^{k-1})$ , there exists an isomorphism  $L_X \cup \tilde{X}_{k-1}^1 \rightarrow L_Y \cup \tilde{Y}_{k-1}^1$  which maps the vertices  $x_{k-1}^1, \dots, x_{k-1}^{k-1}$  to the vertices  $y_{k-1}^1, \dots, y_{k-1}^{k-1}$  respectively, and the pair  $(L_Y \cup \tilde{Y}_{k-1}^1, \tilde{Y}_{k-1}^1)$  is cyclically 2-maximal, where  $L_Y$  is a minimal path in  $Y_{k-1}$  which connects  $y_{k-1}^{k-1}$  and  $\tilde{Y}_{k-1}^1$ . Obviously, in the graph  $Y_{k-1}$  there are no cyclic 2-extensions of the graph  $\tilde{Y}_{k-1}^1$ . If there are no cyclic 2-extensions of the graph  $L_X \cup \tilde{X}_{k-1}^1$  in  $X_{k-1}$ , then the graphs  $\tilde{X}_{k-1}^1 \cup L_X$  and  $\tilde{Y}_{k-1}^1 \cup L_Y$  are  $(k, k-1, 1)$ -regular equivalent in  $(X_{k-1}, Y_{k-1})$ . In the next round, Duplicator exploits the strategy SF. If there is a cyclic 2-extension of the graph  $L_X \cup \tilde{X}_{k-1}^1$  in  $X_{k-1}$ , then  $d_{X_{k-1}}(x_{k-1}^{k-1}, \tilde{X}_{k-1}^1) = 2$ . Moreover, by the property of  $(k, k-2)$ -equivalence of the graphs  $\tilde{X}_{k-2}^1, \tilde{Y}_{k-2}^1$  in  $(X_{k-2}, Y_{k-2})$ , the only path with length 2 which does not coincide with  $L_X$  and connects the vertex  $x_{k-1}^{k-1}$  with some vertex of the graph  $\tilde{X}_{k-1}^1$  satisfies the following property. Its terminal vertex (distinct from  $x_{k-1}^{k-1}$ ) either is not one of the vertices  $x_{k-1}^1, \dots, x_{k-1}^{k-2}$ , or equals one of the terminal vertices of  $L_X$ . Obviously, in the  $k$ -th round, if Spoiler chooses a vertex from one of the graphs  $L_X \cup \tilde{X}_{k-1}^1, L_Y \cup \tilde{Y}_{k-1}^1$ , then Duplicator wins by choosing the image of  $x_k^k$  under an isomorphism of the graphs. If Spoiler chooses a vertex outside these graphs which is adjacent to at most one vertex of  $x_k^1, \dots, x_k^{k-1}$ , then Duplicator has a winning strategy by Property 2) of the graph  $Y_k$ . Obviously, there exist at most two vertices in  $\{x_k^1, \dots, x_k^{k-1}\}$  which are adjacent to  $x_k^k$ . Finally, if the vertex  $x_k^k$  is adjacent to two vertices of  $x_k^1, \dots, x_k^{k-1}$ , then Duplicator chooses the vertex with degree 2 from either the path  $L_X$ , or the path  $L_Y$ , and wins.

#### 4.3.7 Strategy $S_{r+1}^1$

Let after the  $r$ -th round,  $r \in \{2, \dots, k-2\}$ , there exist induced subgraphs  $\tilde{X}_r^1$  and  $\tilde{Y}_r^1$  of  $X_r$  and  $Y_r$  respectively such that the following properties hold. The graph  $\tilde{Y}_r^1$  is cyclically  $2^{k-r}$ -maximal,  $x_r^1, \dots, x_r^r \in V(\tilde{X}_r^1)$ ,  $y_r^1, \dots, y_r^r \in V(\tilde{Y}_r^1)$ , there exists an isomorphism  $\varphi : \tilde{X}_r^1 \rightarrow \tilde{Y}_r^1$  which maps the vertices  $x_r^1, \dots, x_r^r$  to the vertices  $y_r^1, \dots, y_r^r$  respectively. Equalities  $\tilde{X}_r^1 = \tilde{X}_{r-1}^1 \cup L_X$ ,  $\tilde{Y}_r^1 = \tilde{Y}_{r-1}^1 \cup L_Y$



hold, where  $\tilde{X}_{r-1}^1, \tilde{Y}_{r-1}^1$  are graphs which have one common vertex with paths  $L_X$  and  $L_Y$  respectively,  $\varphi|_{\tilde{X}_{r-1}^1} : \tilde{X}_{r-1}^1 \rightarrow \tilde{Y}_{r-1}^1$  is an isomorphism, the vertices  $x_r^1, \dots, x_r^{r-1}$  are in  $V(\tilde{X}_{r-1}^1)$ , the vertices  $x_r^r$  and  $y_r^r$  are terminal vertices of paths  $L_X$  and  $L_Y$  and are not from  $V(\tilde{X}_{r-1}^1)$  and  $V(\tilde{Y}_{r-1}^1)$  respectively. Finally, there exists the only cyclic  $2^{k-r}$ -extension  $C_X \cup \tilde{X}_r^r$  of the graph  $\tilde{X}_r^r$ , where  $C_X$  is a path with length  $l \in [2^{k-r-1}, 2^{k-r})$  which connects the vertex  $x_r^r$  with some not terminal vertex  $x$  of the path  $L_X$ . Moreover,  $l + e(L_X) = 2^{k-r+1}$  and  $d_{X_r}(x, x_r^r) + l = 2^{k-r}$ .

In the  $r+1$ -th round,  $r \in \{1, \dots, k-2\}$ , Spoiler chooses a vertex  $x_{r+1}^{r+1}$ . If  $X_{r+1} = X_r$ , then set  $\tilde{X}_{r+1}^1 = \tilde{X}_r^1, \tilde{Y}_{r+1}^1 = \tilde{Y}_r^1$ . Otherwise, set  $\tilde{X}_{r+1}^1 = \tilde{Y}_r^1, \tilde{Y}_{r+1}^1 = \tilde{X}_r^1$  and rename  $L_X := L_Y, L_Y := L_X$ .

Let  $x_{r+1}^{r+1} \in V(\tilde{X}_{r+1}^1)$ . Duplicator chooses the vertex  $y_{r+1}^{r+1} = \varphi(x_{r+1}^{r+1})$ , if  $X_{r+1} = X_r$ . Duplicator chooses the vertex  $y_{r+1}^{r+1} = \varphi^{-1}(x_{r+1}^{r+1})$ , if  $X_{r+1} = Y_r$ . There are no cyclic  $2^{k-r-1}$ -extensions of the graph  $\tilde{Y}_r^1$  in  $Y_r$ . There is a cyclic  $2^{k-r-1}$ -extension of the graph  $\tilde{X}_r^1$  if and only if  $l = 2^{k-r-1}$  (moreover, the number of such extensions does not exceed one). Suppose that the last equality holds.

Let  $x_{r+1}^{r+1} \in \tilde{X}_{r+1} \setminus L_X$ . Set  $\tilde{X}_{r+1}^1 = \tilde{X}_{r-1}^1, \tilde{Y}_{r+1}^1 = \tilde{Y}_{r-1}^1$ , if  $X_{r+1} = X_r$ , and  $\tilde{X}_{r+1}^1 = \tilde{Y}_{r-1}^1, \tilde{Y}_{r+1}^1 = \tilde{Y}_{r-1}^1$ , otherwise. Set  $\tilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \emptyset), \tilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\}, \emptyset)$ . Obviously,  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) = d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2) = 2^{k-r} + 2^{k-r-1} > 2^{k-r-1}$  and, moreover, there are no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  in  $X_{r+1}$ , there are no cyclic  $2^{k-r-1}$ -extension of the graphs  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  in  $Y_{r+1}$ .

Let  $x_{r+1}^{r+1} \in L_X$ .

If  $x_{r+1}^{r+1}$  and the terminal vertex of the path  $L_X$  from  $\tilde{X}_{r+1}^1$  are at a distance less than  $2^{k-r}$ , then denote a minimal path which connects  $x_{r+1}^{r+1}$  and the vertex from the intersection of  $L_X$  and  $\tilde{X}_{r+1}^1$  by  $\tilde{L}_X$ . Rename  $\tilde{X}_{r+1}^1 := \tilde{X}_{r-1}^1 \cup \tilde{L}_X, \tilde{Y}_{r+1}^1 := \varphi(\tilde{X}_{r-1}^1 \cup \tilde{L}_X)$ , if  $X_{r+1} = X_r$ , and  $\tilde{X}_{r+1}^1 := \tilde{Y}_{r-1}^1 \cup \tilde{L}_X, \tilde{Y}_{r+1}^1 := \varphi^{-1}(\tilde{Y}_{r-1}^1 \cup \tilde{L}_X)$ , otherwise. Set  $\tilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \emptyset), \tilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\}, \emptyset)$ . Obviously,  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) = d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2) > 2^{k-r-1}$ . Moreover, there are no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  in  $X_{r+1}$  and no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  in  $Y_{r+1}$ .

If  $x_{r+1}^{r+1}$  and the terminal vertex of the path  $L_X$  from  $\tilde{X}_{r+1}^1$  are at the distance  $d \geq 2^{k-r}$ , then denote a minimal path connecting  $x_{r+1}^{r+1}$  and the terminal vertex of the path  $L_X$  which is not from  $\tilde{X}_{r+1}^1$  by  $\tilde{L}_X$ . Rename  $\tilde{X}_{r+1}^1 := \tilde{X}_{r-1}^1, \tilde{Y}_{r+1}^1 := \tilde{Y}_{r-1}^1$  and set  $\tilde{X}_{r+1}^2 = \tilde{L}_X, \tilde{Y}_{r+1}^2 = \varphi(\tilde{L}_X)$ , if  $X_{r+1} = X_r$ . Rename  $\tilde{X}_{r+1}^1 := \tilde{Y}_{r-1}^1, \tilde{Y}_{r+1}^1 := \tilde{X}_{r-1}^1$  and set  $\tilde{X}_{r+1}^2 = \tilde{L}_X, \tilde{Y}_{r+1}^2 = \varphi^{-1}(\tilde{L}_X)$ , otherwise. Obviously,  $d_{X_{r+1}}(\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2) = d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2) \geq 2^{k-r} > 2^{k-r-1}$ . Moreover, if  $d > 2^{k-r}$ , then there are no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  in  $X_{r+1}$  and no cyclic  $2^{k-r-1}$ -extensions of the graphs  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  in  $Y_{r+1}$ .

In all the considered cases, ordered tuples  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  and  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  are  $(k, r+1, 2)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Thus, in the next round Duplicator exploits the strategy SF.

If in the last case  $d = 2^{k-r}$ , then in the  $r+2$ -th round Spoiler chooses a vertex  $x_{r+2}^{r+2}$  and, next, Duplicator exploits the strategy which is described in Section 4.3.8.

Finally, let  $l > 2^{k-r-1}$ . Then the graphs  $\tilde{X}_{r+1}^1$  and  $\tilde{Y}_{r+1}^1$  are  $(k, r+1, 1)$ -regular equivalent. Thus, in the  $r+2$ -th round Duplicator exploits the strategy SF.

If  $x_{r+1}^{r+1} \notin V(\tilde{X}_{r+1}^1)$  but  $x_{r+1}^{r+1}$  is in the (only) cyclic  $2^{k-r}$ -extension of the graph  $\tilde{X}_{r+1}^1$ , then denote a minimal path in  $X_{r+1}$  which connects  $x_{r+1}^{r+1}$  and  $x_{r+1}^{r+1}$  by  $\tilde{X}_{r+1}^2$ . Rename  $\tilde{X}_{r+1}^1 := \tilde{X}_{r-1}^1, \tilde{Y}_{r+1}^1 := \tilde{Y}_{r-1}^1$ , if  $X_{r+1} = X_r$ , and  $\tilde{X}_{r+1}^1 := \tilde{Y}_{r-1}^1, \tilde{Y}_{r+1}^1 := \tilde{X}_{r-1}^1$ , otherwise. By Property 2), in  $Y_{r+1}$  there exists a vertex  $y_{r+1}^{r+1}$  and a path  $\tilde{Y}_{r+1}^2$  such that  $d_{Y_{r+1}}(y_{r+1}^{r+1}, \tilde{Y}_{r+1}^1) = d_{Y_{r+1}}(\tilde{Y}_{r+1}^2, \tilde{Y}_{r+1}^1) > 2^{k-r-1}$  and the following properties hold. The graphs  $\tilde{X}_{r+1}^2$  and  $\tilde{Y}_{r+1}^2$  are isomorphic, there exists the respective isomorphism which maps the vertex  $x_{r+1}^{r+1}$  to the vertex  $y_{r+1}^{r+1}$  and in  $Y_{r+1}$  there are no cyclic  $2^{k-r-1}$ -extensions of the graph  $\tilde{Y}_{r+1}^2$ . If  $d_{X_{r+1}}(x_{r+1}^{r+1}, x_{r+1}^{r+1}) < 2^{k-r-1}$ , then, obviously, in  $X_{r+1}$  there are no cyclic  $2^{k-r-1}$ -extensions of

$\tilde{X}_{r+1}^2$ . The ordered tuples  $\tilde{X}_{r+1}^1, \tilde{X}_{r+1}^2$  and  $\tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^2$  are  $(k, r+1, 2)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Thus, in the  $r+2$ -th round Duplicator exploits the strategy SF. If  $d_{X_{r+1}}(x_{r+1}^r, x_{r+1}^{r+1}) = 2^{k-r-1}$ , then in the  $r+2$ -th round Spoiler chooses a vertex  $x_{r+2}^{r+2}$  and, next, Duplicator exploits the strategy which is described in Section 4.3.8.

Finally, let  $x_{r+1}^{r+1} \notin V(\tilde{X}_{r+1}^1)$  and  $x_{r+1}^{r+1}$  be not from a cyclic  $2^{k-r}$ -extension of the graph  $\tilde{X}_{r+1}^1$ . If  $d_{X_{r+1}}(x_{r+1}^{r+1}, \tilde{X}_{r+1}^1) \leq 2^{k-r-1}$ , then rename  $L_X$  in the following way:  $L_X$  is a minimal which connects the vertex  $x_{r+1}^{r+1}$  and some vertex of the graph  $\tilde{X}_{r+1}^1$ . Obviously, in  $X_{r+1}$  there are no cyclic  $2^{k-r-1}$ -extensions of  $\tilde{X}_{r+1}^1 \cup L_X$ . By Property 2), in  $Y_{r+1}$  there exists a vertex  $y_{r+1}^{r+1}$  such that  $d_{Y_{r+1}}(\tilde{Y}_{r+1}^1, y_{r+1}^{r+1}) = d_{X_{r+1}}(\tilde{X}_{r+1}^1, x_{r+1}^{r+1})$  and the following properties hold. There exists an isomorphism  $L_X \cup \tilde{X}_{r+1}^1 \rightarrow L_Y \cup \tilde{Y}_{r+1}^1$  which maps the vertices  $x_{r+1}^1, \dots, x_{r+1}^{r+1}$  to the vertices  $y_{r+1}^1, \dots, y_{r+1}^{r+1}$  respectively and the pair  $(L_Y \cup \tilde{Y}_{r+1}^1, \tilde{Y}_{r+1}^1)$  is cyclically  $2^{k-r-1}$ -maximal, where  $L_Y$  is a minimal path which connects the vertex  $y_{r+1}^{r+1}$  and the graph  $\tilde{Y}_{r+1}^1$  in  $Y_{r+1}$ . Obviously, the graphs  $\tilde{X}_{r+1}^1 := \tilde{X}_{r+1}^1 \cup L_X$  and  $\tilde{Y}_{r+1}^1 := \tilde{Y}_{r+1}^1 \cup L_Y$  are  $(k, r+1, 1)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Therefore, in the  $r+2$ -th round Duplicator exploits the strategy SF. If, finally,  $d_{X_{r+1}}(x_{r+1}^{r+1}, \tilde{X}_{r+1}^1) > 2^{k-r-1}$ , then denote the only (if it exists) cyclic  $2^{k-r-1}$ -extension of the graph  $(\{x_{r+1}^{r+1}\}, \emptyset)$  by  $\tilde{X}_{r+1}^2$  (if there are no such extensions, then set  $\tilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \emptyset)$ ). The inequality  $d_{X_{r+1}}(\tilde{X}_{r+1}^2, \tilde{X}_{r+1}^1) > 2^{k-r-1}$  holds. By Property 2), in  $Y_{r+1}$  there exist a vertex  $y_{r+1}^{r+1}$  and a subgraph  $\tilde{Y}_{r+1}^2$  such that  $d_{Y_{r+1}}(\tilde{Y}_{r+1}^2, \tilde{Y}_{r+1}^1) = 2^{k-r-1} + 1$ , there exists an isomorphism  $\tilde{X}_{r+1}^2 \rightarrow \tilde{Y}_{r+1}^2$  which maps the vertex  $x_{r+1}^{r+1}$  to the vertex  $y_{r+1}^{r+1}$  and there are no cyclic  $2^{k-r-1}$ -extensions of the graph  $\tilde{Y}_{r+1}^2$  in  $Y_{r+1}$ . Obviously, the ordered tuples  $\tilde{X}_{r+2}^1, \tilde{X}_{r+2}^2$  and  $\tilde{Y}_{r+2}^1, \tilde{Y}_{r+2}^2$  are  $(k, r+1, 2)$ -regular equivalent in  $(X_{r+1}, Y_{r+1})$ . Therefore, in the  $r+2$ -th round Duplicator exploits the strategy SF.

#### 4.3.8 The next round strategy

If  $X_{r+2} = X_{r+1}$ , then set  $\tilde{X}_{r+2}^1 = \tilde{X}_{r+1}^1$ ,  $\tilde{X}_{r+2}^2 = \tilde{X}_{r+1}^2$ ,  $\tilde{Y}_{r+2}^1 = \tilde{Y}_{r+1}^1$ ,  $\tilde{Y}_{r+2}^2 = \tilde{Y}_{r+1}^2$ . Otherwise, set  $\tilde{X}_{r+2}^1 = \tilde{Y}_{r+1}^1$ ,  $\tilde{X}_{r+2}^2 = \tilde{Y}_{r+1}^2$ ,  $\tilde{Y}_{r+2}^1 = \tilde{X}_{r+1}^1$ ,  $\tilde{Y}_{r+2}^2 = \tilde{X}_{r+1}^2$ . Denote an isomorphism  $\tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2 \rightarrow \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2$  which maps the vertices  $x_{r+2}^1, \dots, x_{r+2}^{r+1}$  to the vertices  $y_{r+2}^1, \dots, y_{r+2}^{r+1}$  respectively by  $\varphi$ . If  $x_{r+2}^{r+2} \in V(\tilde{X}_{r+2}^1)$ , then Duplicator chooses the vertex  $y_{r+2}^{r+2} = \varphi(x_{r+2}^{r+2})$ . If  $r = k-2$ , then Duplicator wins. If  $r < k-2$ , then, obviously, in  $X_{r+2}$  there are no cyclic  $2^{k-r-2}$ -extensions of the graphs  $\tilde{X}_{r+2}^1, \tilde{X}_{r+2}^2$ , in  $Y_{r+2}$  there are no  $2^{k-r-2}$ -extensions of the graphs  $\tilde{Y}_{r+2}^1, \tilde{Y}_{r+2}^2$ . Thus, the ordered tuples  $\tilde{X}_{r+2}^1, \tilde{X}_{r+2}^2$  and  $\tilde{Y}_{r+2}^1, \tilde{Y}_{r+2}^2$  are  $(k, r+2, 2)$ -regular equivalent in  $(X_{r+2}, Y_{r+2})$ . Therefore, in the  $r+3$ -th round Duplicator exploits the strategy SF.

If the vertex  $x_{r+2}^{r+2}$  is in the only cyclic  $2^{k-r-1}$ -extension of the graph  $\tilde{X}_{r+2}^2$ , then denote a path with minimal length which satisfies the following properties by  $\tilde{L}_X$ . Its terminal vertices coincide with the terminal vertices of the path  $\tilde{X}_{r+2}^2$  and the vertex  $x_{r+2}^{r+2}$  is in  $V(\tilde{L}_X)$ . Obviously, there exists an isomorphism  $\tilde{\varphi} : \tilde{X}_{r+2}^1 \cup L_X \rightarrow Y_{r+2}^1 \cup Y_{r+2}^2$  which maps the vertices  $x_{r+2}^1, \dots, x_{r+2}^{r+2}$  to the vertices  $y_{r+2}^1, \dots, y_{r+2}^{r+2}$ . Therefore, if  $r = k-2$ , then Duplicator wins. If  $r < k-2$ , then, obviously, the ordered tuples  $\tilde{X}_{r+2}^1, \tilde{X}_{r+2}^2 := \tilde{L}_X$  and  $\tilde{Y}_{r+2}^1, \tilde{Y}_{r+2}^2$  are  $(k, r+2, 2)$ -regular equivalent in  $(X_{r+2}, Y_{r+2})$ . Therefore, in the  $r+3$ -th round Duplicator exploits the strategy SF.

If the vertex  $x_{r+2}^{r+2}$  is not in the cyclic  $2^{k-r-1}$ -extension of the graph  $\tilde{X}_{r+2}^2$  and  $d_{X_{r+2}}(x_{r+2}^{r+2}, \tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2) \leq 2^{k-r-2}$ , then denote a minimal path which connects  $x_{r+2}^{r+2}$  and  $\tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2$  by  $\tilde{L}_X$ . Obviously, in  $X_{r+2}$  there are no cyclic  $2^{k-r-2}$ -extensions of the graph  $\tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2 \cup \tilde{L}_X$ . By property 2), in  $Y_{r+2}$  there is a vertex  $y_{r+2}^{r+2}$  such that  $d_{Y_{r+2}}(\tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2, y_{r+2}^{r+2}) = d_{X_{r+2}}(\tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2 \cup \tilde{L}_X, x_{r+2}^{r+2})$  and the following properties hold. There exists an isomorphism  $\tilde{L}_X \cup \tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2 \rightarrow \tilde{L}_Y \cup \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2$  which

maps the vertices  $x_{r+2}^1, \dots, x_{r+2}^{r+2}$  to the vertices  $y_{r+2}^1, \dots, y_{r+2}^{r+2}$  respectively and the pair  $(\tilde{L}_Y \cup \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2, \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2)$  is cyclically  $2^{k-r-2}$ -maximal, where  $\tilde{L}_Y$  is a minimal path which connects  $y_{r+2}^{r+2}$  and  $\tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2$  in  $Y_{r+2}$ . If  $r = k - 2$ , then Duplicator wins. If  $r < k - 2$ , then, obviously, the graphs  $\tilde{X}_{r+2}^1 := \tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2 \cup \tilde{L}_X$  and  $\tilde{Y}_{r+2}^1 := \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2 \cup \tilde{L}_Y$  are  $(k, r + 2, 1)$ -regular equivalent in  $(X_{r+2}, Y_{r+2})$ . Therefore, in the  $r + 3$ -th round Duplicator exploits the strategy SF. Finally, if the vertex  $x_{r+2}^{r+2}$  is not in the cyclic  $2^{k-r-1}$ -extension of the graph  $\tilde{X}_{r+2}^2$  and  $d_{X_{r+2}}(x_{r+2}^{r+2}, \tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2) > 2^{k-r-2}$ , then denote the only (if it exists) cyclic  $2^{k-r-2}$ -extension of the graph  $(\{x_{r+2}^{r+2}\}, \emptyset)$  by  $\tilde{X}_{r+2}^3$  (if there are no such extensions, then set  $\tilde{X}_{r+2}^3 = (\{x_{r+2}^{r+2}\}, \emptyset)$ ). Obviously,  $d_{X_{r+2}}(\tilde{X}_{r+2}^3, \tilde{X}_{r+2}^1 \cup \tilde{X}_{r+2}^2) > 2^{k-r-2}$ . By Property 2), in  $Y_{r+2}$  there are a vertex  $y_{r+2}^{r+2}$  and a subgraph  $\tilde{Y}_{r+2}^3$  such that  $d_{Y_{r+2}}(\tilde{Y}_{r+2}^3, \tilde{Y}_{r+2}^1 \cup \tilde{Y}_{r+2}^2) = 2^{k-r-2} + 1$ , there exists an isomorphism  $\tilde{X}_{r+2}^3 \rightarrow \tilde{Y}_{r+2}^3$  which maps the vertex  $x_{r+2}^{r+2}$  to the vertex  $y_{r+2}^{r+2}$ , and there are no cyclic  $2^{k-r-2}$ -extensions of the graph  $\tilde{Y}_{r+2}^3$ . If  $r = k - 2$ , then Duplicator wins. If  $r < k - 2$ , then, obviously, the ordered tuples  $\tilde{X}_{r+2}^1, \tilde{X}_{r+2}^2, \tilde{X}_{r+2}^3$  and  $\tilde{Y}_{r+2}^1, \tilde{Y}_{r+2}^2, \tilde{Y}_{r+2}^3$  are  $(k, r + 2, 3)$ -regular equivalent in  $(X_{r+2}, Y_{r+2})$ . Therefore, in the  $r + 3$ -th round Duplicator exploits the strategy SF.

## 5 Extended law

Theorem 3 can be extended in the following way.

**Theorem 10** *Let  $k > 3$ ,  $b$  be arbitrary natural numbers. Moreover, let  $\frac{a}{b}$  be an irreducible positive fraction,  $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$ . Denote  $\nu = \max\{1, 2^{k-1} - b\}$ . If  $a \in \{\nu, \nu + 1, \dots, 2^{k-1}\}$ , then  $\alpha \notin S_k^2$ .*

A proof of the theorem is nearly the same as the proof of Theorem 8 from [14], therefore, we do not give here a detailed proof. The idea is the following. As Duplicator has a winning strategy in the game  $\text{EHR}(G, H, k)$  for all pairs of graphs  $(G, H)$  such that  $G, H \in \mathcal{S}$  (see Section 4.3.3), then by Theorem 9 it is sufficient to prove that for any  $\alpha$  from the statement of Theorem 10 there exists  $\varepsilon$  such that  $P(G(n, p) \in \mathcal{S}) \rightarrow 1$  as  $n \rightarrow \infty$  for any  $p \in [n^{-\alpha-\varepsilon}, n^{-\alpha+\varepsilon}]$  (see the proof of Theorem 8 from [14]).

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